I. K. Gujral Punjab Technical University

Refresher

Subject : `Mathematics - IV`

Discrete Mathematics

B.tech CSE 4th SEM

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~ Syllabus ~

PART-A

1. **Sets, Relations and Functions :** Introduction, Combination of Sets, ordered pairs, proofs of general identities of sets, relations, operations on relations, properties of relations and functions, Hashing Functions, equivalence relations, compatibility relations, partial order relations.

2. **Rings and Boolean Algebra :** Rings, Subrings, morphism of rings ideals and quotient rings. Euclidean domains Integral domains and fields Boolean Algebra direct product morphisms Boolean sub-algebra Boolean Rings Application of Boolean algebra (Logic Implications, Logic Gates, Karnaugh-map)

3. **Combinatorial Mathematics :** Basic counting principles Permutations and combinations Inclusion and Exclusion Principle Recurrence relations, Generating Function, Application.

PART-B

4. **Monoids and Groups :** Groups Semigroups and monoids Cyclic semigraphs and submonoids, Subgroups and Cosets. Congruence relations on semigroups. Morphisms. Normal subgroups. Dihedral groups.

5. **Graph Theory :** Graph-Directed and undirected, Eulerian chains and cycles, Hamiltonian chains and cycles Trees, Chromatic number Connectivity, Graph coloring, Plane and connected graphs, Isomorphism and Homomorphism. Applications.

Q 4. Differentiate between an ordered and unordered partition of a finite set. **(PTU, May 2003)**

Solution. Let A be any non-empty set and $A_1, A_2, \ldots\ldots A_n$ are non-empty subsets of A s.t.

1. $A_1 \cup A_2 \ldots\ldots \cup A_n = A$
2. $A_i \cup A_j = \phi \ \forall \ i \neq j$

Then $\{A_1, A_2, \ldots\ldots A_n\}$ is called partition of A

e.g : Let X = {1, 2, 3, ... 8, 9}.

(a) [{1, 3, 6}, {2, 8}, {5, 7, 9}]
is not a partition because $4 \in X$ but 4 does not belong to any of the subsets.

(b) [{1, 5, 7}, {2, 4, 8, 9}, {3, 5, 6}]
is not a partition of X because {1 ,5, 7 } & {3, 5, 6} are not disjoint

(c) [{2, 4, 5, 8}), {1, 9}, {3, 6, 7}]
is a partition of X.

(d) [{1, 2, 7}, {3, 5}, {4, 6, 8, 9}]
is a partition of X.

Q 5. State inclusion and exclusion principle. **(PTU, Dec. 2007, 2003)**

OR

Write down the inclusion and exclusion principle on sets. **(PTU, Dec. 2009)**

Solution. For counting the elements of $A \cup B$ first of all we count the elements of A suppose those are n (A). Then we count the elements of B suppose those are n(B). Now every elements of $A \cap B$ is counted twice, once in A and once in B

Hence $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

or

As $(A-B) \cap (A \cap B) = (A \cap B^c) \cap (A \cap B) = A \cap (B^c \cap B) = A \cap \phi = \phi$

$(A-B) \cup (A \cap B) = A \cap (B^c \cup B) = A \cap \cup = A$

$\therefore$ A is the disjoint union of A–B and $A \cap B$ *i.e.* $n(A) = n(A-B) + n(A \cap B)$ .....(1)

Similarly $n(B) = n(B-A) + n(A \cap B)$ .....(2)

Now $A \cup B$ is the disjoint union of A–B, B–A and $A \cap B$

$\therefore$ $n(A \cup B) = n(A-B) + n(B-A) + n(A \cap B)$

$n(A \cup B) = n(A-B) + n(A \cap B) + n(B-A) + n(A \cap B)$

$= n(A) + n(B) - n(A \cap B)$

Q 6. What is a power set? **(PTU, Dec. 2003)**

Solution. The set which Contains all the possible subsets of given set A and it is denoted by P (A)

e.g. : If $A = \{1,2,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Q 7. Define set with examples. **(PTU, May 2004)**

Solution. A well defined collection of objects. The objects that make up a set are called its elements or members sets. Are usually denoted by capital letters and their elements are usually

denoted by small letters. The statement 'a is an element of set A' can be written as $a \in A$ and 'a is not on element of set A' is written as $a \notin A$.

The following are the some examples of sets.

(a) The collection of vowels in English alphabets. This set contains five elements a, e, i, o, u
(b) The collection of first four prime numbers. This set contains elements 2, 3, 5, 7
(c) The collection of all states in India is a set.
(d) The collection of good cricket players of India is not a set because good players is vague and is not well defind.

There are two ways to represent a set.

(i) Tabular or Roaster form : In this form we list the elements separated by commas, within braces { }.

e.g 1. The set of even nautral number less than 10 in tabular form is given by (2,4,6,8)
e.g 2. The set of vowels in English alphabets in tabular form is given by (a,e,i,o,u)

(ii) Set Builder form : In this from, set is described by a characterizing property P(x) of its elements x. and it is given by { x : P (x) holds}

e.g 1. The set of even natural numbers less than 10 in set builder form is given by

$\{x : x < 10, x = 2n, n \in N\}$

e.g 2. The set (1,2,3,4,5) can be written as

$\{x : x \in N, x \leq 5\}$

Q 8. If A = {1, 2, 4, 5}, B {a, b, c, f} and C = {a, 5} are three given sets. Compute (A ∪ C). (A ∪ C) × B. **(PTU, Dec. 2004)**

Solution. Given $A = \{1, 2, 4, 5\}$; $B = \{a, b, c, f\}$; $C = \{a, 5\}$

$A \cup C = \{1, 2, 4, 5, a\}$

Now $(A \cup C) \times B = \{(1, a), (1, b), (1, c), (1, f), (2, a), (2, b), (2, c), (2, f), (4, a), (4, b), (4, c) (4, f), (5, a), (5, b), (5, c), (5, f), (a, a), (a, b), (a, c), (a, f)\}$.

Q 9. Describe {3, 5, 7, 9, ... 77, 79} in a set builder notation. **(PTU, May 2005)**

Solution. In set builder form the given set is represented as $A = \{(2n + 1) ; n \in N, 1 \leq n \leq 39\}$.

Q 10. Let A = {+, –} and B = {00, 01, 10, 11}. Find A × B. **(PTU, May 2005)**

Solution. A = {+, –} and B = {00, 01, 10, 11}

$\therefore$ A × B = {(+, 00), (+, 01), (+, 10), (+, 11), (–, 00), (–, 01), (–, 10), (–, 11)}.

Q 11. How many subsets of {1, 2, 3, ... 10} contain at least 7 elements? **(PTU, May 2005)**

Solution. Let A = {1, 2, 3, ... 10} so it has 10 distinct elements.

We know that, if S be a finite set with n distinct elements then for $0 \leq r \leq n$, the total number of ways in which r elements can be selected out of n elements is ${}^nC_r$

$\therefore$ Total no. of subsets of A containing r elements = ${}^nC_r$

$\therefore$ Required no. of subsets of A = ${}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

$$= \frac{10!}{7!3!} + \frac{10!}{8!2!} + 10 + 1$$

$$= 120 + 45 + 11 = 176.$$

Q 12. Describe the set of even integers in the set-builder rotation. **(PTU, Dec. 2005)**

Solution. I_e = Set of even integers = {0, ±2, ±4, ±6, ±8 , ...}
In set-builder form I_e = {2n ; n □∈ I}
Where, I = set of integers.

Q 13. How many subsets to of {1, 2, 3, ...9} contain at least 5 elements? **(PTU, Dec. 2005)**

Solution. Let A = {1, 2, ... 9}
for $0 \le r \le n$, as we know that,
The total number of ways in which r-elements can be chosen out of n elements = ^nC_r
∴ Total number of subsets of A containing r-elements = ^9C_r
[∵ A has 9 distinct elements]
∴ Required number of subsets of A contains atleast
5-elements $= {}^9C_5 + {}^9C_6 + {}^9C_7 + {}^9C_8 + {}^9C_9$

$$= \frac{9!}{5!\,4!} + \frac{9!}{6!\,3!} + \frac{9!}{7!\,2!} + 9 + 1$$

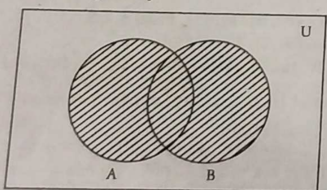
$$= 126 + 84 + 36 + 10$$
$$= 256.$$

Q 14. Define union and intersection of two sets A and B. **(PTU, May 2006)**

Solution. Union of Two Sets : Let A and B be two sets then the set consisting of elements which either belong to A or to B (or to both) is called union of A and B. It is written as A∪B

Thus $A \cup B = \{x : x \in A \text{ or } x \in B\}$
clearly $x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$
and $x \notin AUB \Rightarrow$ $x \notin A$ and $x \notin B$

Venn Diagram for A∪B is given by

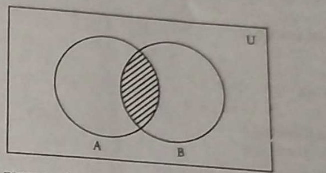


Here the shaded portion represents A∪B clearly from figure $A \subseteq A \cup B$ and $B \subseteq A \cup B$
e.g. If A = {1,2}, B = {3,4,5,2}
Here A∪B = {1,2,3,4,5}

Intersection of two Sets : Let A and B be two sets Then the set consisting of elements which belong to both A and B is called intersection of two sets and It is written as A∩B.

Thus $A \cap B = \{x : x \in A \text{ and } x \in B\}$
Clearly $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
and $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

Venn Diagram for A∩B is given by



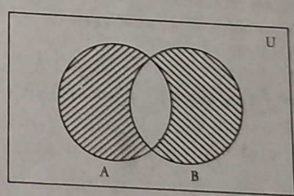
The shaded portion represents A∩B
clearly from figure ; $A \cap B \subseteq A$, $A \cap B \subseteq B$
e.g. If A = {2,3}, B = {1,2,3,4}
Here A∩B = {2,3}

Q 15. Discuss symmetric difference of two sets with examples. **(PTU, May 2007)**

Solution. Let A and B be any two sets. Then the set (A–B)∪(B–A) defines symmetric diff. of two sets. and It is written as AΔB or A⊕B

$A \Delta B = \{x : x \notin A \cap B\}$
If A = {1,2,3} and B = {3,4,5}
AΔB = {1,2,4,5}

Venn Diagram for AΔB is given by



The shaded portion represents AΔB.

Q 16. Find the power set P (A) of A = {1, 2, 3}. **(PTU, Dec. 2007)**

Solution. The set which Contains all the possible subsets of given set A and it is denoted by P (A)

e.g. : If $A = \{1,2,3\}$
then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Q 17. Let A, B, C be arbitrary sets. Prove that $A - (B - C) = A - (B \cup C)$. **(PTU, Dec. 2003)**

Solution. Let $x \in (A - B) - C$ be arbitrary element

$\Rightarrow x \in A - B$ and $x \notin C$
$\Rightarrow x \in A$ and $x \notin B$ and $x \notin C$
$\Rightarrow x \in A$ and $(x \notin B \cup C)$
$\Rightarrow x \in A - (B \cup C)$
$\therefore \quad (A - B) - C \subseteq A - (B \cup C)$ ...... (1)
$\forall\, y \in A - (B \cup C) \Rightarrow y \in A$ and $y \notin B \cup C$
$\Rightarrow y \in A$ and $(y \notin B$ and $y \notin C)$
$\Rightarrow (y \in A$ and $y \notin B)$ and $y \notin C$
$\Rightarrow y \in A - B$ and $y \notin C$
$\Rightarrow y \in (A - B) - C$
$\therefore \quad A - (B \cup C) \subseteq (A - B) - C$ .....(2)

From (1) and (2) ; we get

$$A - (B \cup C) = (A - B) - C.$$

Q 18. Prove :

(a) The complement of the union of two sets equals the intersection of the complements.

(b) The complement of the intersection of two sets equals the union of the complements. **(PTU, May 2004)**

OR

Write down DeMorgan's law for set. **(PTU, May 2011 ; Dec. 2009)**

Solution. Mathematically, we write

(a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$

Proof : (a) $\forall\, x \in (A \cup B)' \Rightarrow x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$
$\Rightarrow \quad (A \cup B)' \subseteq A' \cap B'$ .....(1)
$\forall\, y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B' \Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow \quad y \notin A \cup B \Rightarrow y \in (A \cup B)'$
$\Rightarrow \quad (A' \cap B') \subseteq (A \cup B)'$ .....(2)

From (1) and (2), we get

$$A' \cap B' = (A \cup B)'$$

(b) $\forall\, x \in (A \cap B)' \Rightarrow x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$
$\Rightarrow x \in A'$ or $x \in B' \Rightarrow x \in A' \cup B'$
$\Rightarrow \quad (A \cap B)' \subseteq A' \cup B'$ .....(1)
Also $\quad \forall\, y \in A' \cup B' \Rightarrow y \in A'$ or $y \in B' \Rightarrow y \notin A$ or $y \notin B$
$\Rightarrow \quad y \notin A \cap B \Rightarrow y \in (A \cap B)' \Rightarrow A' \cup B' \subseteq (A \cap B)'$ .....(2)

From (1) and (2), we have

$$(A \cap B)' = A' \cup B'.$$

Q 19. Let A, B and C be sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **(PTU, May 2005)**

Solution. $\forall\, (x, y) \in A \times (B \cap C)$
$\Rightarrow x \in A$ and $y \in B \cap C$
$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$
$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$
$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$
$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$
$\Rightarrow A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ .....(1)
$\forall\, (a, b) \in (A \times B) \cap (A \times C)$
$\Rightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C$
$\Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$
$\Rightarrow a \in A$ and $(b \in B$ and $b \in C)$
$\Rightarrow a \in A$ and $b \in B \cap C$
$\Rightarrow (a, b) \in A \times (B \cap C)$
$\Rightarrow (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ .....(2)

From (1) and (2) ; we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Q 20. Prove that $A \cup (B - A) = A \cup B$. **(PTU, Dec. 2005)**

Solution. $\forall\, x \in A \cup (B - A) \Leftrightarrow x \in A$ or $x \in B - A$
$\Leftrightarrow x \in A$ or $(x \in B$ and $x \notin A)$
$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \notin A)$ [Using $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$]
$\Leftrightarrow x \in A$ or $x \in B$
$\Leftrightarrow x \in A \cup B$
$\Leftrightarrow A \cup (B - A) = A \cup B$.

Q 21. Find the number of subsets of a set S containing n elements. (PTU, May 2007)

Solution. Let S be any finite set containing n distinct elements.

$0 \le r \le n$, Now we know that the total number of ways in which r elements can be chosen out of n element in ${}^nc_r$. therefore total subsets of S containing r elements is ${}^nc_r$.

No. of subsets of S containing no element is ${}^nc_0$
No. of subsets of S containing one element is ${}^nc_1$
No. of subsets of S containing two elements is ${}^nc_2$

....

No. of subsets of S containing n elements is ${}^nc_n$
$\therefore$ Total Number of subsets of $S = {}^nc_0 + {}^nc_1 ..... + {}^nc_n = 2^n$ (Sum of binomial co-effs.)

Q 22. If A and B are two subsets of a universal set then prove that $A - B = A \cap \overline{B}$. **(PTU, May 2008)**

Solution.

$$\forall x \in A-B \Rightarrow x \in A \text{ and } x \notin B \Rightarrow x \in A \text{ and } x \in B^c \Rightarrow x \in A \cap B^c \Rightarrow A-B \subseteq A \cap B^c \quad \text{.....(1)}$$

$$\forall y \in A \cap B^c \Rightarrow y \in A \text{ and } y \in B^c \Rightarrow y \in A \text{ and } y \notin B \Rightarrow y \in A-B \Rightarrow A \cap B^c \subseteq A-B \quad \text{.....(2)}$$

From (1) and (2) we get

$$A-B = A \cap B^c$$

Q 23. In a class of 60 boys, 45 boys play cards and 30 boys play carrom. How many boys play both games? How many play cards only and how many play carrom only? **(PTU, May 2008)**

Solution. Let A and B be the sets of boys play cards and play carrom

$\therefore$ $n(A) = 45$; $n(B) = 30$; $n(A \cup B) = 60$

(i) Number of boys play both games $= n(A \cap B)$

as we know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 45 + 30 - 60 = 15$

(ii) No. of boys play cards only $= n(A \cap B^C)$

$= n(A) - n(A \cap B)$

$= 45 - 15 = 30$

(iii) No. of boys play carrom only $= n(B \cap A^C)$

$= n(B) - n(B \cap A)$

$= 30 - 15 = 15$

Q 24. For $A = \{1, 2, \{1, 3\}, \phi\}$, determine the following sets. **(PTU, May 2009)**
(a) $A - \{1\}$ (b) $A - \phi$ (c) $A - \{\phi\}$ (d) $A - \{1, 2\}$.

Solution. (a) Given $A = \{1, 2, \{1, 3\}, \phi\}$

$A - \{1\} = \{2, \{1, 3\}, \phi\}$

(b) $A - \phi = \{1, 2, \{1, 3\}\}$

(c) $A - \{\phi\} = \{1, 2, \{1, 3\}, \phi\}$

(d) $A - \{1, 2\} = \{\{1, 3\}, \phi\}$.

Q 25. How many integers between 1 and 300 (inclusive) are
(a) Divisible by at least one of 3, 5, 7?
(b) Divisible by 3 and 5, not by 7.
(c) Divisible by 5 but neither by 3 or 7? **(PTU, Dec. 2009)**

Ans. Let A, B, C are the set of integers between 1 and 300 which are divisible by 3, 5 and 7 respectively.

$$\therefore \quad n(A) = n\left(\frac{300}{3}\right) = 100 \text{ ; } n(A \cap B) = n\left(\frac{300}{3.5}\right) = 20$$

$$n(B) = n\left(\frac{300}{5}\right) = 60 \text{ ; } n(A \cap C) = n\left(\frac{300}{3.7}\right) = 14$$

$$n(C) = n\left(\frac{300}{7}\right) = 42 \text{ ; } n(B \cap C) = n\left(\frac{300}{5.7}\right) = 8$$

and $\quad n(A \cap B \cap C) = n\left(\frac{300}{3.5.7}\right) = 2$

We know that,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 100 + 60 + 42 - 20 - 8 - 14 + 2$

$= 162$

(a) Req. no. of integers which are divisible by atleast one of 3, 5, 7

$= 300 - n(A^C \cap B^C \cap C^C)$

$= 300 - n[(A \cup B \cup C)^C]$

$= 300 - \{300 - n(A \cup B \cup C\} = 162.$

(b) Req. no. of integers $= n(A \cap B \cap C^C) = n(A \cap B) - n(A \cap B \cap C)$

$= 20 - 2 = 18.$

(c) Req. no. of integers $= n(B \cap A^C \cap C^C) = n(B \cap (A \cup C)^C)$

$= n(B) - n[B \cap (A \cup C)]$

$= n(B) - n\{(B \cap A) \cup (B \cap C)\}$

$= n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$

$= 60 - 20 - 8 + 2 = 34.$

Q 26. In a class of 60 boys, 45 boys play cards and 30 play carom. How many boys play both games? How many plays cards only and how many plays caroms only? **(PTU, May 2010)**

Solution. Let $n(A)$ = No. of students who plays cards = 45

$n(B)$ = No. of boys who plays carom = 30

Also, $\quad n(A \cup B) = 60$

We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ No. of boys who play both games

$= n(A \cap B) = 45 + 30 - 60 = 15$

No. of boys who plays cards only $= n(A \cap B^C)$

$= n(A) - n(A \cap B)$

$= 45 - 15 = 30$

No. of boys who plays carom only $= n(B \cap A^C)$

$= n(B) - n(B \cap A)$

$= 30 - 15 = 15.$

Q 27. Show that how the set operations of union and intersection may be defined for classes of sets. **(PTU, May 2003)**

Solution. Case-I. If the classes of sets contains a finite number of collection of sets

Let $A_1, A_2, \ldots\ldots A_n$ are the finite number of sets.

Then operations of union and intersection on there sets may be defined as follows :

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \ldots\ldots \cup A_n = \{x : x \in A_i \text{ for some } A_i, 1 \le i \le n\}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \ldots\ldots \cup A_n = \{x : x \in A_i \text{ for each } A_i, 1 \le i \le n\}$$

So union consists of those elements which belongs to atleast one set of the collection and intersection consists of those elements which belongs to each set of the collection.

Case-II. If the classes of sets contains a arbitrary number of collection of sets.

Let A_i be any arbitrary collection of sets.

Then $\cup A_i = A_1 \cup A_2 \cup \ldots. = \{x : x \in A_i \text{ for some } A_i\}$

$\cap A_i = A_1 \cap A_2 \cap \ldots. = \{x : x \in A_i \text{ for each } A_i\}$

Q 28. Prove that if A, B, C, D are arbitrary sets, then $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. **(PTU, Dec. 2003)**

Solution. Let (x,y) be any arbitrary element of $(A{\times}B) \cap (C{\times}D)$

$(x,y) \in (A{\times}B)\cap(C{\times}D) \Rightarrow (x,y) \in A{\times}B$ and $(x,y) \in C{\times}D$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in C$ and $y \in D)$

$\Rightarrow$ $(x \in A$ and $x \in C)$ and $(y \in B$ and $y \in D)$

$\Rightarrow$ $x \in A \cap C$ and $y \in B \cap D$

$\Rightarrow$ $(x,y) \in (A \cap C) \times B \cap D \Rightarrow (A{\times}B)\cap(C{\times}D) \subseteq (A \cap C) \times (B \cap D)$ .....(1)

Let (s,t) be any arbitrary element of $(A \cap C) \times (B \cap D)$

$\Rightarrow$ $s \in A \cap C$ and $t \in B \cap D$

$\Rightarrow$ $(s \in A$ and $s \in C)$ and $(t \in B$ and $t \in D)$

$\Rightarrow$ $(s \in A$ and $t \in B)$ and $(s \in C$ and $t \in D)$

$\Rightarrow$ $(s,t) \in A{\times}B$ and $(s,t) \in C{\times}D$

$\Rightarrow$ $(s,t) \in (A{\times}B) \cap (C{\times}D)$

$\Rightarrow$ $(A \cap C) \times B \cap D \subseteq (A{\times}B) \cap (C{\times}D)$ ....(2)

From (1) & (2) ; we have

$(A \cap C) \times (B \cap D) = (A{\times}B) \cap (C{\times}D)$.

Q 29. Suppose that 100 of the 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose 65 study French, 45 study German, 42 study Russian, 20 study French and German, 25 study French and Russian, 15 study German and Russian. Find the number of students studying all subjects. Also find the number of students studying taking exactly one subject. **(PTU, May 2004)**

Solution. Let F, G and R be the sets of students studying French, German and Russian respectively.

$\therefore$ n(F)=65, n(G)=45, n(R)=42 , $n(F \cap G)=20$, $n(F \cap R)=25$, $n(G \cap R)=15$, $n(F \cup G \cup R)=100$

(i) No. of students studying all subjects $= n(F \cap G \cap R)$

we know that

$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$

$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(F \cap G \cap R)$

$\Rightarrow$ $n(F \cap G \cap R) = 100 - 92 = 8$

(ii) Number of students studying French alone $= n(F \cap G^c \cap R^c)$

$= n(F \cap (G \cup R)^c)$

$= n(F) - n(F \cap (G \cup R))$

$= n(F) - n[(F \cap G) \cup (F \cap R)]$

$= n(F) - [n(F \cap G) + n(F \cap R) - n[(F \cap G) \cap (F \cap R)]]$ $[\because n(A-B) = n(A) - n(A \cap B)]$

$= n(F) - n(F \cap G) - n(F \cap R) + n(F \cap G \cap R)$

$= 65 - 20 - 25 + 8 = 28$

Number of students studying Russian alone $= n(R \cap F^c \cap R^c)$

$= n(R) - n(R \cap F) - n(G \cap R) + n(R \cap F \cap G)$

$= 42 - 25 - 15 + 8 = 10$

Number of students studying German alone $= n(G \cap F^C \cap R^C)$

$= n(G) - n(G \cap F) - n(G \cap R) + n(G \cap F \cap R)$

$= 45 - 20 - 15 + 8 = 18$

$\therefore$ Number of students studying excatly one subject

$= 28 + 18 + 10 = 56$

Q 30. If A, B are any two sets, then $A \cap B = B \cap A$. **(PTU, May 2006)**

Solution. Let x be any arbitrary element of $A \cup B$

$\Rightarrow$ $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow$ $x \in B$ and $x \in A \Rightarrow x \in B \cap A$

$\Rightarrow$ $A \cap B \subseteq B \cap A$

Similarly we can prove that

$B \cap A \subseteq A \cap B$

Hence $A \cap B = B \cap A$.

Q 31. Prove that :

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $(A \cap B)' = A' \cup B'$. **(PTU, May 2007, 2006)**

Solution. (a) $\forall (x, y) \in A \times (B \cap C)$ .....(1)

$\Rightarrow x \in A$ and $y \in B \cap C$

$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow (x,y) \in (A \times B) \cap (A \times C)$ ....(2)

From (1) and (2) ; we get

$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ....(*)

$\forall (a, b) \in (A \times B) \cap (A \times C)$ ....(3)

$\Rightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C$

$\Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow a \in A$ and $(b \in B$ and $b \in C)$

$\Rightarrow a \in A$ and $(b \in B \cap C)$

$\Rightarrow (a, b) \in A \times (B \cap C)$ ....(4)

From (3) and (4) ; we get

$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ....(5)

$\therefore$ From (*) and (5) ; we have

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $\forall\, x \in (A \cap B)' \Rightarrow x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow \quad x \in A'$ or $x \in B' \Rightarrow x \in A' \cup B'$

$\Rightarrow \quad (A \cap B)' \subseteq A' \cup B'$ ....(1)

Also $\quad \forall\, y \in A' \cup B' \Rightarrow y \in A'$ or $y \in B' \Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow \quad y \notin A \cap B \Rightarrow y \in (A \cap B)' \Rightarrow A' \cup B' \subseteq (A \cap B)'$ ....(2)

From (1) and (2), we have

$(A \cap B)' = A' \cup B'$

Q 32. If A is any set, then $(A')' = A$. **(PTU, Dec. 2007)**

Solution. Let x be any arbitrary element of $(A')'$

$\Rightarrow x \in (A')' \Rightarrow x \notin A' \Rightarrow x \in A$

$\Rightarrow \quad (A')' \subseteq A$ ....(1)

Let y be any arbitrary element of A

$\Rightarrow y \in A \Rightarrow y \notin A' \Rightarrow y \in (A')'$

$\therefore \quad A \subseteq (A')'$ ....(2)

From (1) and (2) ; we get

$A = (A')'$ [If $A \subseteq B$ and $B \subseteq A$ then $A = B$]

Q 33. Prove that $(A \cap B) \cap C = A \cap (B \cap C)$, for any sets A, B, C. **(PTU, Dec. 2007)**

Solution. $\forall\, x \in (A \cap B) \cap C \Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow x \in A$ and $x \in B \cap C \Rightarrow x \in A \cap (B \cap C)$

$\Rightarrow (A \cap B) \cap C \subseteq A \cap (B \cap C)$ ....(1)

$\forall\, y \in A \cap (B \cap C) \Rightarrow y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow (y \in A$ and $y \in B)$ and $(y \in C)$

$\Rightarrow y \in A \cap B$ and $y \in C$

$\Rightarrow y \in (A \cap B) \cap C$

$\Rightarrow A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ....(2)

From (1) and (2) ; we get

$(A \cap B) \cap C = A \cap (B \cap C)$.

Q 34. Let X = {1, 2, ... 8, 9}. Determine whether or not each of the following is a partition of X.

(a) [{1, 3, 6}, {2, 8}, {5, 7, 9}]

(b) [{1, 5, 7}, {2, 4, 8, 9}, {3, 5, 6}] **(PTU, May 2011)**

Solution. Let S be any non – empty set Then the collection of sets $\{A_i\}$ of non empty set S is a partition if

(i) Every element of S is belongs to one of the set A_i

(ii) The sets of $\{A_i\}$ are mutually disjoint *i.e.* if $A_i \neq A_j$ Then $A_i \cap A_j = \phi$

(a) is not a partition because $4 \in X$ but 4 does not belong to any of the subsets.

(b) is not a partition of X because {1 ,5, 7 } & {3, 5, 6} are not disjoint.

Q 35. If A and B are any two sets then prove that

$$A \cup B = A \cap B \Leftrightarrow A = B$$

(PTU, Dec. 2014)

Solution. Given $\quad A \cup B = A \cap B$ ....(1)

T.P $\quad A = B$

Let $x \in A$ be any arbitrary element

i.e. $\quad x \in A$ or $x \in B \Rightarrow x \in A \cup B = A \cap B$

$\Rightarrow \quad x \in A$ and $x \in B \;\therefore A \subseteq B$ [using (1)]

Let $x \in B$ be any arbitrary element

$\therefore \quad x \in B$ or $x \in A \Rightarrow x \in B \cup A$

$\Rightarrow \quad x \in A \cup B$ [$\because A \subseteq A \cup B$]

$\Rightarrow \quad x \in A \cap B$ [$\because A \cup B = B \cup A$]

$\Rightarrow \quad x \in A$ & $x \in B$ [$\because$ eq of (1)]

$\Rightarrow \quad x \in A \Rightarrow B \subseteq A$

$\therefore \quad A \subseteq B$ & $B \subseteq A \Rightarrow A = B$

Converse $A = B$ T.P $A \cup B = A \cap B$

Now $\quad A \cup B = A \cup A = A$ [$\because$ idempetent law & given]

& $\quad A \cap B = A \cap A = A$ [$\because$ idempetent law & given]

$\therefore \quad A \cup B = A \cap B$

□□□

Chapter 2

Relation and Function

QUESTION-ANSWERS

Q 1. Find the numbers of relation from A = {a, b, c} to B = {1, 2}. (PTU, Dec. 2002)

Solution. Let A and B be two sets consisting of m and n elements. Since Relation $R \subseteq A \times B$ and $O(A \times B) = O(A)\,O(B) = mn$, So the total Number of Subsets of $A \times B$ is 2^{mn}. Hence there are 2^{mn} relation from A to B.

Hence total number of relations from A to B = $2^{O(A \times B)}$

$= 2^{O(A).O(B)} = 2^{3.2} = 64$

Q 2. Find the formula for the inverse of g (x) = x² – 1. (PTU, Dec. 2002)

Solution. Let $g(x) = y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \pm\sqrt{y-1}$

$\Rightarrow \quad g^{-1}(y) = \pm\sqrt{y-1}$ is the required formula.

Q 3. Define function the following with examples. (PTU, May 2011, 2009, 2004)

Solution. If X and Y are two non-empty sets Then the mapping or correspondence f from X to Y is a rule which assigns every element x of X to a unique elementy y of Y. Here y is called image of x under f. and the element x is called preimage of y. we write it as y = f (x) and f is a function form X to Y denoted by $f: X \to Y$.

Q 4. Define relation the following with examples. (PTU, May 2011, 2004)

Solution. We shall understood relation with the help of following example. Let A and B be the set of all males and females in the royal family of Dashrath kingdom.

A = [Dashrath, Ram, Laxman, Bharat, Shatrughan]

B = [Kaushalya, Kaikai, Sumitra, Sita, Urmila, Mandvi, Shrukirti]

If we have the relation R "was husband of " between the elements of A and B.

So Dashrath R Kaushalya, Dashrath R Kaikai, Dashrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi, Shatrughan R Shrutkirti.

So In other words we can write

R = [(Dashrath, Kaushalya), (Dashrath, Kaikai), (Dashrath Sumitra), (Ram, Sita) (Laxman, Urmila),(Bharat, Mandvi) (Shatrughan, Shrutkirti)]

clearly $R \subseteq A \times B$.

Let us take up an another example where A = [2, 4, 6] and B = [4, 8, 12, 18] and R be the relation " is divisor of "

So 2R4, 2R8, 2R12, 2R18, 4R4, 4R8, 4R12, 6R12, 6R18

R = [(2,4), (2,8), (2,12), (2,18), (4,4), (4,8), (4,12), (6,12), (6,18)]

The above two examples leads to the def. of relation as follows Let A and B be two Non–empty sets Then a relation or binary relation R from A to B is subset of $A \times B$ or Relation R from A to B is the set of all ordered pair (x,y) where $x \in A$ and $y \in B$ for which the statement xRy or $(x,y) \in R$ is true or false.

Q 5. Define the following concept giving one example of each :

(i) onto function

(ii) Antisymmetric relation or set

(iii) Union of two sets

(iv) Partial order relation. (PTU, May 2011 ; Dec. 2011, 2004)

Solution. (i) Onto function : Let us define a function $f : A \to B$ for any $y \in B$ $\exists$ some element $x \in A$ s.t $y = f(x)$.

Then f is said to be onto

e.g. Let $f : I \to I$ defined by $f(x) = x \ \forall\ x \in I$

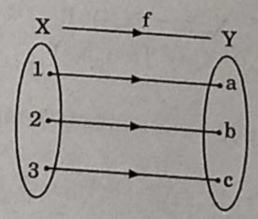
where I be the set of integers.

Here f (1) = 1, f (0) = 0, f (–1) = –1 and so on ∴ f is onto.

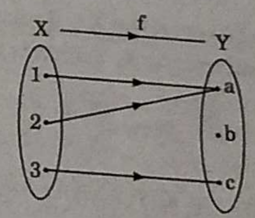
Further let $f : N \to R$ defined by $f(x) = x^2 \ \forall\ x \in N$

Here f is not onto, as corresponding to every negative real number, we have no pre-images.

Graphically onto and into function is given as under :



onto function



into function

(ii) Antisymmetric relation : A relation R on set A is said to be antisymmetric iff $\forall\, a,b \in A$, aRb, bRa Then a= b

e.g. 1. The Relation R on set A natural numbers N defined by xRy iff $x \le y \ \forall\, x,y \in N$

It is antisymmetric as $\forall\, x,y \in N$, $xRy \Rightarrow x \le y$ and $yRx \Rightarrow y \le x$ *i.e.* $x \le y, y \le x \Rightarrow x = y$

e.g. 2. The Relation R on set of natural numbers N defined by $\forall\, x,y \in N$, xRy iff x divides y

This relation is antisymmetric as $\forall\, x,y \in N$

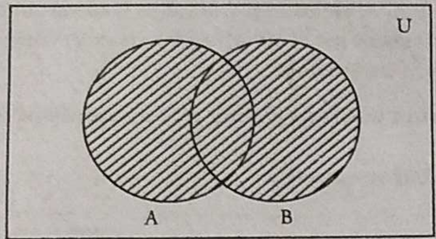
$$\left.\begin{array}{l} xRy \Rightarrow x / y \\ yRx \Rightarrow y / x \end{array}\right] \Rightarrow x = y$$

(iii) Union of Two Sets : Let A and B be two sets then the set consisting of elements which either belong to A or to B (or to both) is called union of A and B. It is written as $A \cup B$

Thus $\quad A \cup B = \{x : x \in A \text{ or } x \in B\}$

clearly $\quad x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$

Venn Diagram for $A \cup B$ is given by



Here the shaded portion represents $A \cup B$ clearly from figure $A \subseteq A \cup B$ and $B \subseteq A \cup B$

e.g. $\quad$ If $A = \{1,2\}$, $B = \{3,4,5,2\}$

Here $\quad A \cup B = \{1,2,3,4,5\}$

(iv) Partial order relation : A Relation R on set A is said to be partial order relation iff

(i) R is reflexive

(ii) R is antisymmetric

(iii) R is transitive

Then the set A with partial order Relation R is said to be poset.

e.g : Let P (A) be the power set of A and R be a Relation on P (A) defined by (X,Y)∈R iff X ⊆ Y Then R is partial order relation on P(A).

Solution.

(i) Reflexive : As every set X is a subset of itself $\therefore X \subseteq X \Rightarrow (X,X) \in R$ $\therefore$ R is reflexive

(ii) Antisymmetric : $\forall$ X, Y ∈P(A) s.t (X,Y) ∈R , (Y,X)∈R Then X=Y

$$\text{Now} \left.\begin{matrix}(X,Y) \in R \Rightarrow X \subseteq Y \\ (Y,X) \in R \Rightarrow Y \subseteq X\end{matrix}\right] \Rightarrow X = Y \therefore \text{R is anti-symmetric}$$

(iii) Transitive : $\forall$ X, Y, Z∈P(A) . we have (X,Y) ∈R, (Y,Z)∈R Then (X,Z) ∈R

$$\text{Now} \left.\begin{matrix}(X,Y) \in R \Rightarrow X \subseteq Y \\ (Y,Z) \in R \Rightarrow Y \subseteq Z\end{matrix}\right] \Rightarrow (X \subseteq Z) \in R \therefore \text{R is transitive}$$

Hence R is partial order relation on P(A).

Q 6. Let A = B = {1, 2, 3, 4, 5} define function f : A → B such that f is one and onto function. **(PTU, Dec. 2005)**

Solution. Given A = B = {1, 2, 3, 4, 5}

Now function $f : A \to B$ defined by $f(x) = x \ \forall x \in A$

Clearly $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 4$ and $f(5) = 5$

$\therefore$ Different elements of set A have different images in set B $\therefore$ f is one-one.

Also corresponding to every element in B has a pre-image in B $\therefore$ f is onto.

Q 7. Define partial order relation. **(PTU, May 2006)**

Solution. A Relation R on set A is said to be partial order relation iff

(i) R is reflexive

(ii) R is antisymmetric

(iii) R is transitive

Then the set A with partial order Relation R is said to be poset.

Q 8. What are domain, co domain and image of a function? **(PTU, Dec. 2006)**

OR

Define domain and range of a relation. **(PTU, May 2007)**

Solution. Let $f : X \to Y$ is a function Here X is called Domain of f. The Set Y is called codomain of f.

The set of all f- image of X is called range of f.

$$R_f = f(X) = \{f(x) : x \in X\} \subseteq Y$$

e.g 1 : Let $f : I \to Y$ be a function defined by $f(x) = x^2 \ \forall x \in I$

Range of $f = \{0,1,4,9,16,....\} = \{n^2 : n \in I\}$

e.g 2 : Let X = {1,2,3, ...} and Y = {1,2,3, ...}

Let $f : X \to Y$ defined by $f(x) = \begin{cases} 1 ; x \text{ is odd} \\ 0 ; x \text{ is even} \end{cases}$

Here $D_f = N$ and $R_f = \{0,1\}$

Q 9. The relation {(1, 2), (1, 3) (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)} is :

(i) Reflexive $\quad$ **(ii) Transitive**

(iii) Symmetric $\quad$ **(iv) Asymetric** **(PTU, May 2008)**

Solution. As $(2, 2) \notin R$ $\therefore$ relation is not reflexive

Further As $(1, 2) \in R$ but $(2, 1) \notin R$ $\therefore$ relation is not symmetric

Further As $(1, 3), (3, 1) \in R \Rightarrow (1, 1) \in R$

$(1, 3), (3, 3) \in R \Rightarrow (1, 3) \in R$

$(1, 3), (3, 2) \in R \Rightarrow (1, 2) \in R$

$(1, 3), (3, 4) \in R \Rightarrow (1, 4) \in R$

$(1, 1), (1, 4) \in R \Rightarrow (1, 4) \in R$

$(1, 4), (4, 2) \in R \Rightarrow (1, 2) \in R$

$(3, 1), (1, 1) \in R \Rightarrow (3, 1) \in R$

$(3, 1) (1, 4) \in R \Rightarrow (3, 4) \in R$

$(3, 1) (1, 2) \in R \Rightarrow (3, 2) \in R$

$(3, 1), (1, 3) \in R \Rightarrow (3, 3) \in R.$

$\therefore$ Relation is transitive thus Ans. (ii)

Q 10. Reflexive relation. **(PTU, May 2009)**

Solution. Reflexive Relation : Let A be any set

Then R on A is said to be reflexive iff $(a,a) \in R \ \forall \ a \in A$

e.g. 1. Let A = {1,2,3} Then R = {(1,1), (2,2), (3,3), (1,2)} is reflexive relation on A while R_1 = {(1,1), (2,2), (1,3)} is not reflexive as $(3,3) \notin R_1$ where $3 \in A$

e.g. 2. Let A be any Non empty set and P(A) be the power set of A Then R on P(A) is defined as ARB iff $A \subseteq B$
If is a reflexive relation as every set is a subset of itself.($A \subseteq A$)

e.g. 3. Let R be a relation on L (set of lines in a plane) defines as follows $(l_1, l_2) \in R$ iff $l_1 \| l_2$.
Then it is a reflexive relation as every line is parallel to itself.

e.g. 4. If R be a relation on N defined by xRy iff x≥,y Then R be reflexive as every natural number is greater than or equal to itself.

Q 11. Let A = Z⁺, be the set of positive integers and R be the relation on A defined by a R b if and only if there exists a k ∈Z⁺ such that $a = b^k$. Which one of the following belongs to R?

(i) (8, 128) **(ii) (16, 256)**
(iii) (11, 3) **(iv) (169, 13)** **(PTU, Dec. 2008)**

Solution. Given A = Z⁺ be the set of +ve integers and R be the relation on A defined by a Rb iff ∃ a k ∈ Z⁺
s.t. $a = b^k$ as $169 = 13^2$, where k = 2 ∈Z⁺
∴ (169, 13) ∈R ∴ Ans. (iv)

Q 12. Check whether relation of divisibility on the set N of positive integers is an equivalence relation or not? Justify your answer. **(PTU, Dec. 2009)**

Solution. As a/a ∀ a ∈ N ∴ Relation | of divisibility is reflexive
∀ (a, b) ∈ N s.t a/b, it does not means b/a
as 1/2 but 2/1 ∴ Relation of divisibility is not symmetric.
Thus relation defined by divisibility is not an equivalence relation.

Q 13. Define an equivalence relation and give an example of the same.
(PTU, May 2012, 2011, 2010 ; Dec. 2007)

OR

What are the properties for a relation to be equivalence relation?
(PTU, Dec. 2005)

Solution. A relation R is said to be equivalence relation on non-empty set A iff R is reflexive, symmetric and transitive on A.

(i) Reflexive $\forall \ l \in L$, $(l, l) \in R$ as $l \| l$ i.e every time is || to itself.

(ii) Symmetric : $\forall \ l_1, l_2 \in L$ if $(l_1, l_2) \in R$ Then $(l_2, l_1) \in R$

Now $(l_1, l_2) \in R \Rightarrow l_1 \| l_2 \Rightarrow l_2 \| l_1 \Rightarrow (l_2, l_1) \in R$

(iii) Transitive : $\forall l_1, l_2, l_3 \in L$ if $(l_1, l_2) \in R$, $(l_2, l_3) \in R$ Then $(l_1, l_3) \in R$

Now $\left.\begin{array}{l}(l_1, l_2) \in R \Rightarrow l_1 \| l_2 \\ (l_2, l_3) \in R \Rightarrow l_2 \| l_3\end{array}\right] \Rightarrow l_1 \| l_3 \Rightarrow (l_1, l_3) \in R$

Hence R is reflexive, symmetric and transitive on L ∴ R is equivalence relation on L.

Q 14. Suppose g (t) = $t^3 - 2t^2 - 6t - 3$. Find the roots of g (t), assuming g (t) has an integral root. **(PTU, Dec. 2002)**

Solution. The given polynomial is g (t) = $t^3 + 2t^2 - 6t - 3$
Clearly t = –1 satisfies given equation ∴ –1 is the root of eq (1) so t + 1 is a factor of eq. (1). ...... (1)
∴ By synthetic division method, we have

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -6 & -3 \\ & & -1 & 3 & 3 \\ \hline & 1 & -3 & -3 & 0 \end{array}$$

∴ quotient is given by $t^2 - 3t - 3 = 0$ *i.e.* $t = \frac{3 \pm \sqrt{21}}{2}$

∴ roots of g (t) are $-1, \frac{3 \pm \sqrt{21}}{2}$.

Q 15. Show that the relation of being associates in an equivalence relation in a ring R. **(PTU, Dec. 2002)**

Solution. First of all, we define associates in a ring R.
If a, b ≠ 0 ∈ R then a and b are associates iff a/b, b/a
Further let us define a relation of associates 's' on R
as a s b ⇔ a and b are associates *i.e.* a/b, b/a
Now we want to prove that 's' is equivalence relation in R.
Reflexive : As a/a ∴ a is an associates of a
∴ a s a ⇒ s is reflexive.
Symmetric : ∀ a, b ≠ 0 ∈ R s.t a s b ⇒ a is an associate of b
⇒ a/b, b/a ⇒ b/a, a/b ⇒ b and a are associates
therefore b s a ⇒ s is symmetric.
Transitive : Let a and b are associates in R *i.e.* a/b, b/a ..... (1)
also b s c *i.e.* b and c are associates ∴ b/c, c/b ..... (2)
Now a/b, b/c ⇒ a/c
also c/b, b/a ⇒ c/a
∴ a/c, c/a ⇒ a and c are associates.
∴ a s c ∴ s is transitive.
∴ 's' is reflexive, symmetric and transitive.
Thus s is an equivalence relation in a ring R.

Q 16. Define one-one and onto function. Give example. **(PTU, May 2011)**

OR

Define the term injective, subjective and bijective with example. **(PTU, May 2003)**

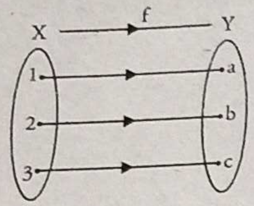
Solution. (i)A function $f : X \to Y$ is said to be one-one or injective if different elements of X have different images.

i.e. $\forall\ x_1, x_2 \in X$ s.t $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

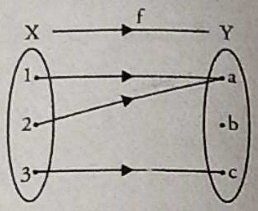
or if $f(x_1) = f(x_2)\ \forall x_1, x_2 \in X \Rightarrow x_1 = x_2$

It is abbreviated as (1–1)

e.g. 1.



f is 1–1 because different elements have different images.



f is not 1–1 because the elements 1,2 ∈ X have same image

e.g. 2. Let $f: N \to N$ defined by $f(x) = x^2\ x \in N$

It is 1–1 because different element have different images.

e.g. 3. Let $f : R \to N$ defined by $f(x) = x^2\ \forall x \in R$

It is not 1–1 (many one) as different elements have same images. ($\because$ 2, –2 have same image 4)

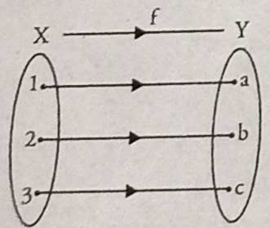
(ii) Onto or Surjective function :

Let $f : X \to Y$ be a function Then f is said to be onto if each element in Y has a pre image in X i.e For onto function we have $f(X) = Y$

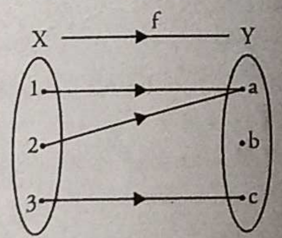
otherwise it is said to be into.

e.g. 1.

(i)



(ii)



(i) f is onto as corresponding to every element in Y has a preimage in. X

(ii) f is not onto as b∈Y we have no image in X.

e.g. 2. Let $f : X \to Y$ be a function defined by $f(x) = x^2\ \forall x \in X$

Where X = N ; Y = R (set of real numbers)

Here f is not onto as corresponding to every negative real numbers we have no preimage

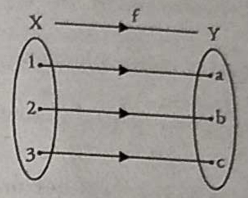
(iii) one–one, onto or Bijection :

A function $f : X \to Y$ is bijective iff it is one–one and onto

i.e. (i) $\forall x, y \in X$ s.t $f(x) = f(y) \Rightarrow x = y$

(ii) $\forall y \in Y\ \exists\ x \in X$ s.t $f(x) = y$

e.g. 1.



Here f is 1–1 as well as onto $\therefore$ It is bijection.

e.g. 2. the function $f : Q \to Q$ defined by $f(x) = 2x+1\ \forall x \in Q$ It is 1–1 and onto

For 1–1 : $\forall x, y \in Q$ s.t $f(x) = f(y) \Rightarrow 2x+1 = 2y+1 \Rightarrow x = y$ $\therefore$ f is 1–1.

onto : $\forall y \in Q$ Then $f(x) = y = 2x+1 \Rightarrow x = \frac{y-1}{2}$

Clearly $\forall y \in Q \exists\ \frac{y-1}{2} \in Q$ s.t $f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1 = y$

$\therefore$ f is onto

$\therefore$ f is 1–1 and onto So f is bijection

Q 17. Let R be the set of all pairs (a, b) where a and b are mathematicians that have been co-authors on a paper.

(a) Prove whether or not R is an equivalence rotation.

(b) Describe the meaning of RoR.

(c) Describe the transitive closure of R. Prove that this is an equivalence relation.

(d) Give an example that shows that R does not necessarily partition a set of mathematicians. **(PTU, May 2004)**

Solution. (a) Here R be a set of all pairs a and b s.t (a, b) ∈ R iff a and b are mathematicians that have been co-authors on a paper.

Clearly R is reflexive as (a, a) ∈R

Further if a and b are mathematician that have been co-authors on a paper then b and a are also co-authors on a paper $\therefore$ R is symmetric.

If a and b co-authors on a paper and are mathematicians and b and c co-authors on a paper. Then it is not necessary that a and c are co-authors on a paper.

$\therefore$ R is not transitive.

Thus R is not an equivalence relation.

(b) Let R be a relation from A to B

Then aRoRb iff for same b ∈ B, we have (a, b) ∈R

Thus RoR means it is the set of all pairs (a, b) where a and b are mathematicians which are co-author on a paper.

(c) Here we need a mathematician c which is a co-author of a and b on a paper.

∴ Transitive closure of R is given as under

R^∞ or R^* = {(a, a), (a, b), (b, a), (b, b), (c, c), (c, b), (b, c), (a, c), (c, a)}

Clearly R* is an equivalence relation as R* is transitive also and by first (a) it is reflexive and symmetric.

(d) Here, the set of all ordered pairs (a, b) gives the set of mathematicians.

Consider the collection S of (a, b) as

S = {[(a, b)], [(b, a)]} since (a, b) and (b, a) are not different.

∴ We can't form a partition on set of mathematicians.

Q 18. Consider the following relations of the set A = (1, 2, 3, 4) defined by,

(i) R = {(1, 1), (1, 2), (1, 3), (3, 3)}

(ii) S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

(iii) T = {(1, 1), (1, 2), (2, 2), (2, 3)}

(iv) φ = Empty relation.

(v) U = Universal relation

Determine whether or not each of the above relations on A is

(a) Reflexive (b) Symmetric (c) Transitive (d) Antisymmetric. **(PTU, May 2004)**

Solution. (a) Reflexive :

(i) Since (2, 2) ∉ R ∴ R is not reflexive

(ii) As (1, 1), (2, 2), (3, 3) ∈S but (4, 4) ∉S ∴ S is not reflexive

(iii) (3, 3) ∉T ∴ T is not reflexive

(iv) Empty relation on a non-empty set A is not reflexive

(v) Universal relation is always reflexive.

(b) Symmetric :

(i) As (1, 2) ∈R but (2, 1) ∉R ∴ R is not symmetric

(ii) S is symmetric as (a, b) ∈S ⇒ (b, a) ∈S

(iii) T is not symmetric as (1, 2) ∈T but (2, 1) ∉T

(iv) φ defined on a non-empty set is always symmetric

(v) U defined on a non-empty set is always symmetric.

(c) Transitive :

(i) If (a, b) ∈R, (b, c) ∈R then (a, c) ∈R ∴ R is transitive.

(ii) Clearly S is transitive

(iii) As (1, 2), (2, 3) ∈T but (1, 3) ∉T ∴ T is not transitive

(iv) φ defined on non-empty set is always transitive.

(v) U is always transitive.

(d) Antisymmetric :

(i) Relation R is antisymmetric if (a, b) ∈R, (b, a) ∈R Then a = b

Clearly R is antisymmetric

(ii) Here (1, 2) ∈S, (2, 1) ∈S but 1 ≠ 2 ∴ S is not antisymmetric.

(iii) Here (1, 2), (2, 1) ∈T but 1 ≠ 2 ∴ T is not antisymmetric.

(iv) Empty relation φ defined on A is not antisymmetric.

(v) Let universal relation U defined an a non empty set A containing atleast two distinct elements is antisymmetric *i.e.* ∀ a, b ∈ A s.t (a, b) ∈R and (b, a) ∈R

Then a = b which is a contradiction to the fact that a and b are distinct ∴ U defined on non-empty set A is not antisymmetric.

Q 19. Give an example of a relation which is :

(i) Neither reflexive nor irreflexive.

(ii) Both symmetric and antisymmetric.

(iii) Both reflexive and symmetric.

(iv) Both reflexive, symmetric and transitive.

(v) Both symmetric and transitive but not reflexive. **(PTU, Dec. 2004)**

Solution. (i) Let A = {a, b, c, d} and let us define a relation on A

R = {(a, b), (b, c), (c, d), (b, b)} as (a, a) ∉ R

∴ R is neither reflexive nor irreflexive.

[∵ If (x, x) ∉R ∀x ∈R then R is irreflexive]

(ii) Let A = {1, 2, 3, 4} and let us define a relation on A

i.e. R = {(1, 1), (2, 2), (3, 3), (4, 4)}

Here ∀ x, y ∈ A s.t (x, y) ∈R Then (y, x) ∈R ⇒ R is symmetric

Further (x, y) ∈R, (y, x) ∈R ⇒ x = y ∀x, y ∈ A

∴ R is antisymmetric.

(iii) Let R be a relation on set of lines in a plane L

defined by $l_1 R_2 l_2$ iff $l_1 \parallel l_2$

as $l_1 \parallel l_1$ ∵ every line is ∥ to itself

∴ (l_1, l_1) ∈R thus R is reflexive.

Further ∀ l_1, l_2 ∈L s.t. $l_1 R l_2 \Rightarrow l_1 \parallel l_2 \Rightarrow l_2 \parallel l_1$

∴ (l_2, l_1) ∈R, thus R is symmetric.

(iv) The relation R on set of integers I defined by xRy ⇒ x – y is divisible by n is an equivalence relation.

(a) Reflexive : ∀x∈I , x–x = 0 =0 .n ⇒ x–x is divisible by n ⇒ (x,x)∈R or xRx

(b) Symmetric : ∀x,y∈I s.t xRy ⇒ x–y is divisible by n

⇒ x–y = np ∀p∈I ⇒ y–x = n(–p) ⇒ $n|_{y-x}$ ⇒ y–x is divisble by n ⇒ yRx

(c) Transitive : ∀x,y,z∈I s.t xRy and yRz T.P xRz

Now xRy ⇒ x–y = np ∀p∈I .....(1)

yRz ⇒ y–z = nq ∀q∈I .....(2)

on adding (1) & (2), we have

⇒ x–z = n(p+q) where p+q∈I

⇒ n/x–z ⇒ x–z is divisible by n

⇒ xRz

Hence R is an equivalence relation on I.

(v) Let $A = \{1, 2, 3\}$ and let us define a relation on A

i.e. $R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$

Clearly it is symmetric and transitive but not reflexive as $(3, 3) \notin R$.

Q 20. Define composition of relations with example. **(PTU, May 2007)**

Solution. Let R be a Relation from A to B and S be the Relation From B to C Then the Relation SoR from A to C is defined by a SoRc iff for some $b \in B$ we have a Rb and bSc, for $a \in A$ & $c \in C$

e.g. Let $A = \{1,2,3,4\}$, $R = \{(1,1), (1,2), (2,3), (2,4), (3,4), (4,1), (4,2)\}$

and $S = \{(3,1), (4,4), (2,3), (2, 4), (1,1), (1,4)\}$

Then, $SoR = \{(1,1), (1,4), (1,3), (2,1), (2,4), (3,4), (4,1), (4,4), (4,3)\}$

Q 21. Prove that "congruence modulo H, $a \equiv b \pmod H$" is an equivalence relation in G. **(PTU, Dec. 2007)**

Solution. The Relation Congruence modulom means

if $aRb \Rightarrow a \equiv b \pmod H \Rightarrow a-b$ is divisible by H.

(i) Reflexivity : $\forall a \in I$ s.t $a-a = 0 = 0.H \Rightarrow a-a$ is divisible by $H \Rightarrow a \equiv a \pmod H$

(ii) Symmetry : $\forall a,b \in I$ s.t $a \equiv b \pmod H$ T.P $b \equiv a \pmod H$ As $a \equiv b \pmod H \Rightarrow H \mid a-b$

$\Rightarrow a-b$ is divisible by H $\therefore a-b = Hp$ $\forall p \in I \Rightarrow b-a = H(-p) \Rightarrow H \mid b-a \Rightarrow b-a$ divisible by H

$\therefore b \equiv a \pmod H$

(iii) Transitivity : $\forall a,b,c \in I$ s.t $aRb \Rightarrow a \equiv b \pmod H$ and $bRc \Rightarrow b \equiv c \pmod H$ T.P aRc

Now $a \equiv b \pmod H \Rightarrow H \mid a-b \Rightarrow a-b = Hp$ where $p \in I$

$b \equiv c \pmod H \Rightarrow H \mid b-c \Rightarrow b-c = Hq$ where $q \in I$

on adding, we get

$\Rightarrow$ $a-c = H(p+q) \Rightarrow H \mid a-c \Rightarrow a-c$ is divisible by H

$\Rightarrow$ $a \equiv c \pmod H \Rightarrow aRc$

Hence the relation congruence mod H is an equivalence relation on I

Q 22. Consider the function $f : N \to N$, where N is the set of natural numbers defined by $f(n) = n^2 + n + 1$. Show that the function f is one-one but not onto. **(PTU, Dec. 2008)**

Solution. $f : N \to N$ defined by $f(n) = n^2 + n + 1$ $\forall n \in N$

$\forall n, m \in N$, let $f(n) = f(m)$

$\Rightarrow$ $n^2 + n + 1 = m^2 + m + 1$

$\Rightarrow n^2 - m^2 + n - m = 0$

$\Rightarrow (n - m)(n + m + 1) = 0$

$\Rightarrow$ $n = m$ $[\because n = -m - 1 \notin N]$

$\therefore \forall n, m \in N$ s.t $f(n) = f(m) \Rightarrow n = m$

$\therefore$ f is one-one

Now $1 \in N$ s.t $f(n) = 1 \Rightarrow n^2 + n + 1 = 1 \Rightarrow n = 0, -1 \notin N$

$\therefore \forall 1 \in N \ \exists$ no $n \in N$ s.t $f(n) = 1$

$\therefore$ f is not onto.

Q 23. Write all possible relations from $A = \{0\}$ to $B = \{1, 2\}$. **(PTU, May 2009)**

Solution. Now we know that $R \subseteq A \times B$

Where, $A \times B = \{(0, 1), (0, 2)\}$

$\therefore$ The possible relations are

ϕ, $\{(0, 1)\}$, $\{(0, 2)\}$, $\{(0, 1), (0, 2)\}$.

Q 24. Give an explicit formula for a function from the set of integers to the set of positive integers i.e.

(a) One to one but not onto.

(b) Onto but not one to one.

(c) One to one and onto.

(d) Neither one to one nor onto.

Solution. (a) Let us define $f : I \to N$ by $f(x) = x^2 + x + 1$ $\forall x \in I$ **(PTU, May 2009)**

Clearly f is 1 – 1 as different elements have different images and $2 \in N \ \exists$ no $x \in I$ s.t. $f(x) = x^2 + x + 1$

$[\because$ if $2 = x^2 + x + 1 \Rightarrow x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} \notin I]$

$\therefore$ f is not onto although it is 1 – 1.

(b) Let us define $f : I \to N$ by $f(x) = \begin{cases} 2 & ; x \text{ is an odd integer} \\ -2 & ; x \text{ is an even integer} \end{cases}$

Clearly f is not one-one but f is onto.

(c) Let us define $f : I \to N$ by $f(x) = x$

Clearly it is 1–1 as different elements having different images also it is onto.

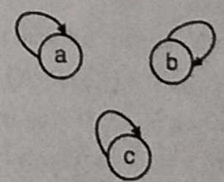
(d) Let us define $f : I \to N$ by $f(x) = x^2$

It is not 1–1, as 2, –2, ... having same images also f is not onto as 2 has no pre image in I.

$\therefore$ f is neither 1–1 and not onto. $[\because 2 = x^2 \Rightarrow x = \pm\sqrt{2} \notin I]$

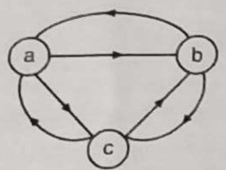
Q 25. Using Graph Representation of a relative how we can identify that a relation is reflexive, symmetric and anti-symmetric. **(PTU, Dec. 2009)**

Solution. Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c)\}$ be an reflexive relation on A



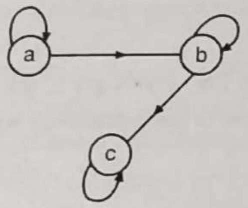
As there is a edge from a to a, b to b, and c to c

Let R = {(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)} be a symmetric relation on A.



As there is a edge from a to b and back b to a similarly there is an edge from b to c, c to a and back to c to b, a to c.

Let R = {(a, a), (b, b), (c, c), (a, b), (b, c)} be an antisymmetric relation on A.



Q 26. Let X = {1, 2, 3, 4, 5, 6, 7} and R {(x, y) | x – y is divisible by 3}. Check whether this equivalence relation or not? Give appropriate reason in support of your answer. (PTU, Dec. 2009)

Solution. Given X = {1, 2, 3, 4, 5, 6, 7}

and R = {(x, y) | x – y is divisible by 3}

(i) Reflexive : $\forall$ x $\in$ X, x – x = 0 = 0.3 $\therefore$ x – x is divisible by 3

i.e. (x, x) $\in$R

(ii) Symmetric : $\forall$x, y $\in$ X s.t (x, y) $\in$R *i.e.* x – y is divisible by 3

$\therefore$ x – y = 3p $\forall$ p $\in$ X

y – x = –3p but –p $\notin$ X as X contains all the natural numbers upto 7.

$\therefore$ (y, x) $\notin$ R

$\therefore$ R is not symmetric.

Hence R is not an equivalence relation on X.

Q 27. Let R be the relation on the set of ordered pairs of positive integers such that (a, b) R (c, d) if and only if a + d = b + c. Show that R is an equivalence relation. (PTU, May 2010)

Solution. (i) Reflexive : (a,b) R (a,b) $\Rightarrow$ a+b = b+a

Which is true as $\forall$ a,b,$\in$N commutative law holds under addition.

(ii) Symmetric : $\forall$(a,b), (c,d) $\in$N×N s.t (a,b) R (c,d) T.P (c,d) R (a,b)

Now (a,b) R (c,d) $\Rightarrow$ a+d = b+c $\Rightarrow$ d+a = c+b $\Rightarrow$ (c,d) R (a,b)

(iii) Transitive : $\forall$(a,b), (c,d) (e,f) $\in$N×N s.t (a,b) R (c,d) and (c,d) R (e,f) Then (a,b) R(e,f)

Now (a,b) R (c,d) $\Rightarrow$ a+d = b+c

(c,d) R (e,f) $\Rightarrow$ c+f = d+e

on adding

$\Rightarrow$ (a+d) +(c+f) = (b+c) + (d+e)

$\Rightarrow$ a+f = b+e $\Rightarrow$ (a,b) R (e,f)

$\therefore$ R is an equivalence relation on N×N as It is reflexive, symmetric and transitive.

Q 28. Prove : A function f : A $\rightarrow$ B has an inverse iff f is bijective. (PTU, Dec. 2002)

Solution.

Let f : X$\rightarrow$Y is invertible T.P f is 1–1, onto Since f is invertible So by def. $\exists$ g : Y$\rightarrow$ X s.t fog = I_Y and gof = I_X .... (*)

First of all we prove that f is one–one

$\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow g[(f(x_1)] = g[(f(x_2)]$

$\Rightarrow$ $(gof)(x_1) = (gof)(x_2) \Rightarrow I_X(x_1) = I_X(x_2)$

$\Rightarrow$ $x_1 = x_2 \Rightarrow$ f is 1–1

Now we prove that f is onto

To each y$\in$Y $\exists$ x$\in$X $\Rightarrow$ g (y) = x

$\Rightarrow$ g [g (y)] = f (x) $\Rightarrow$ fog (y) = f (x) $\Rightarrow I_Y$ y = f (x)

$\Rightarrow$ y = f (x) $\therefore$ f is onto.

Converse : Let f is 1–1, onto T.P f is invertible

Now f is 1–1 and onto So $\forall$ y$\in$Y $\exists$ unique x$\in$X s.t f (x) = y

$\therefore$ we can define a function g : Y $\rightarrow$X s.t g (y) = x iff f (x) = y

Now (fog) (y) = f (g (y)) = f (x) = y $\forall$y$\in$Y $\Rightarrow$ fog = I_Y

(gof) (x) = g (f(x)) = g (y) = x $\forall$x$\in$X $\Rightarrow$ gof = I_X

$\therefore$ f is invertible.

Q 29. Let A = {1, 2, 3, 4} and let r be the relation $\le$ on A. Draw the diagraph of r. (PTU, May 2005)

Solution. Given, A = {1, 2, 3, 4}

$\forall$ a, b $\in$ A s.t a r b $\Rightarrow$ a $\le$ b

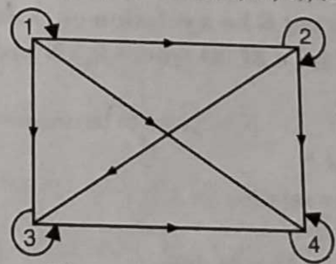
Now 1 $\le$ 1 $\Rightarrow$ (1, 1) $\in$r, 2 $\le$ 2 $\Rightarrow$ (2, 2) $\in$r, 4 $\le$ 4 $\Rightarrow$ (4, 4) $\in$r

1 $\le$ 2 $\Rightarrow$ (1, 2) $\in$r, 2 $\le$ 3 $\Rightarrow$ (2, 3) $\in$r, 3 $\le$ 3 $\Rightarrow$ (3, 3) $\in$r

1 $\le$ 3 $\Rightarrow$ (1, 3) $\in$r, 2 $\le$ 4 $\Rightarrow$ (2, 4) $\in$r

1 $\le$ 4 $\Rightarrow$ (1, 4) $\in$r, 3 $\le$ 4 $\Rightarrow$ (3, 4) $\in$r

$\therefore$ r = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}



Digraph of relation r

Q 30. How many different reflexive, symmetric relation are there on a set with three elements? **(PTU, May 2005)**

Solution. Let the set be denoted by A containing three elements

$\therefore$ A = {a, b, c}

for any a, b $\in$ A s.t a R b $\Rightarrow$ (a, b) $\in$R

Reflexive and symmetric relation with 3 given elements

(i) {(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c), (c,a)}

(ii) {(a, a), (b, b), (c, c), (a, b), (b, a)}

(iii) {(a, a), (b, b), (c, c), (b, c), (c, b)}

(iv) {(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)}

(v) {(c, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)}

(vi) {(a, a), (b, b), (c, c), (b, c), (c, b), (a, c), (c, a)}

(vii) {(a, b), (b, b), (c, c), (a, c), (c, a)}

(viii) {(a, a), (b, b), (c, c)}

So there are eight relations which are reflexive as well as symmetric on set A.

Q 31. Let N = {1, 2, 3, ...} and let be the relation on N × N defined by (a, b) R (c, d) if a + d = b + c. Prove that it is an equivalence relation. **(PTU, May 2008)**

OR

Let R be the relation on the set of ordered pairs of positive integers such that (a, b) R (c, d) if and only if a + d = b + c. Show that R is an equivalence relation. **(PTU, May 2012)**

Solution. (i) Reflexive : (a,b) R (a,b) $\Rightarrow$ a+b = b+a

Which is true as $\forall$ a,b,$\in$N commutative law holds under addition.

(ii) Symmetric : $\forall$ (a,b), (c,d) $\in$N×N s.t (a,b) R (c,d) T.P (c,d) R (a,b)

Now (a,b) R (c,d) $\Rightarrow$ a+d = b+c $\Rightarrow$ d+a = c+b $\Rightarrow$ (c,d) R (a,b)

(iii) Transitive : $\forall$ (a,b), (c,d) (e,f) $\in$N×N s.t (a,b) R (c,d) and (c,d) R (e,f) Then (a,b) R(e,f)

Now (a,b) R (c,d) $\Rightarrow$ a+d = b+c

(c,d) R (e,f) $\Rightarrow$ c+f = d+e

on adding ; we get

$\Rightarrow$ (a+d) +(c+f) = (b+c) + (d+e)

$\Rightarrow$ a+f = b+e $\Rightarrow$ (a,b) R (e,f)

$\therefore$ R is an equivalence relation on N×N as It is reflexive, symmetric and transitive.

Q 32. Let A $\subseteq$ Z and f : A $\rightarrow$ N be a one-one function where Z is a set of integers and N is a set of natural numbers. Let R be a relation on A defined as under :

(x, y) $\in$R is and only if f (y) = kf (x) where k $\in$N prove that R is a partial order relation on A. **(PTU, Dec. 2008)**

Solution. Since A $\subseteq$ Z and f : A $\rightarrow$ N be given to be one-one and R be relation on A defined by (x, y) $\in$R iff f (y) = k f (x) $\forall$ k $\in$ N

To prove : R is partial order relation on A.

i.e. we want to prove that R is

(a) reflexive (b) anti-symmetric (c) transitive

Since f (x) = 1 . f (x), where 1 $\in$ N $\Rightarrow$ (x, x) $\in$ R $\Rightarrow$ R is reflexive

Let (x, y) $\in$R $\Rightarrow$ f (y) = kf (x), k $\in$N ....(1)

and (y, x) $\in$R $\Rightarrow$ f (x) = k'f (y), k' $\in$ N ....(2)

From (1) and (2), we have

f(x) = kk'f (n) $\Rightarrow$ (1 –kk') f (x) = 0 $\Rightarrow$ k = ± 1 = k'

but k', k $\in$N $\therefore$ k = 1 = k' [as f (x) $\neq$ 0, otherwise question has no utility]

$\therefore$ eq (1) gives, f (y) = f (x), as f is one-one

$\therefore$ x = y

Thus R is antisymmetric

Let (x, y) $\in$R $\Rightarrow$ f (y) = kf (x) where k $\in$N

and (y, z) $\in$R $\Rightarrow$ f (z) = k' f (y) where k' $\in$N

$\Rightarrow$ f(z) = kk'f (x) = k'' f (x)

$\Rightarrow$ (x, z) $\in$R [as k, k' $\in$N $\Rightarrow$ kk' = k'' $\in$N]

$\therefore$ R is transitive.

Thus R is reflexive, antisymmetric and transitive.

$\therefore$ R is partial order relation on A.

Q 33. Let A = {1, 2, 3, 4} and let R and S be the relations on A described by

$$M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Use Warshall's algorithm to compute the transitive closure of R $\cup$ S. **(PTU, Dec. 2008)**

Solution. Given,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore$ R = {(1, 4), (3, 2), (4, 3)}

also

$$M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$\therefore$ S = {(1, 1), (1, 2), (2, 2), (3, 3), (4, 2), (4, 4)}

Thus R $\cup$ S = T = {(1, 1), (1, 2), (1, 4), (2, 2), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)}

$$M_T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_T^2 = M_T\, M_T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_T^3 = M_T^2\, M_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_T^4 = M_T^3\, M_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad M_T^{\infty} = M_T \cup M_T^2 \cup M_T^3 \cup M_T^4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

i.e. $T^{\infty} = (R \cup S)^{\infty} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,2), (3,3), (4,2), (4,3), (4,4)\}$

Q 34. How many positive integers not exceeding 500 are divisible by 7 or 11? **(PTU, Dec. 2010)**

Solution. Let A, B denotes the set of integers between 1 and 500 which are divisible by 7 and 11.

i.e. $n(A) = n\left(\frac{500}{7}\right) = 71$; $n(B) = n\left(\frac{500}{11}\right) = 45$

$$n(A \cap B) = n\left(\frac{500}{7.11}\right) = n\left(\frac{500}{77}\right) = 6.$$

$\therefore$ The number of positive integers between 1 and 500 which are divisible by 7 or 11 = $n(A \cup B)$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 71 + 45 - 6 = 120.$

Q 35. Let P, Q and R be three finite sets. Prove that $|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |Q \cap R| + |P \cap Q \cap R|$.

Ans. Let P, Q, R are three finite sets **(PTU, Dec. 2010)**

Then $n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(P \cap R) + n(P \cap Q \cap R)$

proof : Let D be the union of two sets Q and R

i.e. $D = Q \cup R$

$\therefore \quad n(P \cup D) = n(P) + n(D) - n(P \cap D)$

[using Inclusion Exclusion principle for two sets] ..... (1)

Now $n(D) = n(Q \cup R)$

$= n(Q) + n(R) - n(Q \cap R)$ ......(2)

Now $n(A \cap D) = n[P \cap (Q \cup R)] = n[(P \cap Q) \cup (P \cap R)]$

$= n(P \cap Q) + n(P \cap K) - n[(P \cap Q) \cap (P \cap R)]$

$= n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)$ ......(3)

$\therefore$ from (1), (2) and (3); we get

$n(P \cup D) = n(P \cup Q \cup R)$

$= n(P) + n(Q) + n(R) - n(P \cap Q) - n(P \cap R) - n(Q \cap R) + n(P \cap Q \cap R)$

Hence proved.

Q 36. What are the properties of relations? Explain with examples. Find the number of relation from the set A = {a, b, c} to B = {1, 2}. **(PTU, Dec. 2010)**

Ans. (I) Void or Empty Relation : Let A be any set. Then $\phi \subseteq A \times B$ is a relation on A called empty relation on A.

(II) Universal Relation : Let A be any set. Then $A \times A$ is a relation on A called universal relation on A.

(III) Identity Relation : Let A be any set.

Then the relation $I_A = \{(a,a) : (a \in A)\}$ *i.e.* every element of the set related to itself.

e.g. R = {(1,1), (2,2), (3,3), (4,4)} on A = {1,2,3,4} is an identity relation on A. Whereas R_1 = {(1,1), (2,2), (3,3), (4,4), (1,4)} is not an identity relation on A as 1 is related to 1 & 4.

(IV) Reflexive Relation : Let A be any set

Then R on A is said to be reflexive iff $(a,a) \in R \ \forall\ a \in A$

e.g. 1. Let A = {1,2,3} Then R = {(1,1), (2,2), (3,3), (1,2)} is reflexive relation on A while R_1 = {(1,1), (2,2), (1,3)} is not reflexive as $(3,3) \notin R_1$ where $3 \in A$

e.g. 2. Let A be any Non empty set and P(A) be the power set of A Then R on P(A) is defined as ARB iff $A \subseteq B$

If is a reflexive relation as every set is a subset of itself.($A \subseteq A$)

e.g. 3. Let R be a relation on L (set of lines in a plane) defines as follows $(l_1, l_2) \in R$ iff $l_1 \,||\, l_2$.

Then it is a reflexive relation as every line is parallel to itself.

e.g. 4. If R be a relation on N defined by xRy iff $x \geq y$ Then R be reflexive as every natural number is greater than or equal to itself.

(V) Symmetric Relation : A relation R on set 'A' is said to be symmetric iff $\forall\ a,b \in A$ $(a,b) \in R \Rightarrow (b,a) \in R$

e.g. 1. Let R be a Relation on L (set of lines in a plane) defined by $(l_1, l_2) \in R$ iff $l_1 \perp l_2$ Then R be a symmetric relation on L. because if $l_1 \perp l_2$ Then $l_1 \perp l_2$

e.g. 2. Let A = {1,2,3} Then R = {(1,2), (2,1), (1,3), (3,1)} be a symmetric relation on A as $(1,2) \in R$ Then $(2,1) \in R$ and $(1,3) \in R$ Then $(3,1) \in R$

e.g. 3. Let A be any set and P(A) be power set of A Then Relation R on P(A) defined by CRD iff $C \subseteq D$ is not symmetric relation because if CRD $\Rightarrow C \subseteq D \not\Rightarrow D \subseteq C$ *i.e.* DRC

Transitive Relation : A relation R on set A is said to be transitive iff $\forall$ a,b,c$\in$A, $(a,b) \in R$, $(b,c) \in R$ Then $(a,c) \in R$

e.g. 1. The Relation R on power set of 'A' defined by $(C,D) \in R$ iff $C \subseteq D$ is a transitive relation on A

$\because$ If $(C,D) \in R \Rightarrow C \subseteq D$ .....(1) and $(D,E) \in R \Rightarrow D \subseteq E$ .....(2)

From (1) & (2), $C \subseteq D \subseteq E \Rightarrow C \subseteq E \Rightarrow (C,E) \in R$

e.g. 2. The Relation R on L (set of lines in a plane) is defined by $(l_1, l_2) \in R$ iff $l_1 \| l_2$

It is a transitive relation as $(l_1, l_2) \in R, (l_1 \| l_2) \in R$

i.e. $(l_2, l_3) \in R \Rightarrow l_2 \| l_3 \Rightarrow l_1 \| l_3 \Rightarrow (l_1, l_3) \in R$.

e.g. 3. The Relation R on Natural number N defined by xRy $\Rightarrow x \geq y \; \forall$ x, y$\in$N

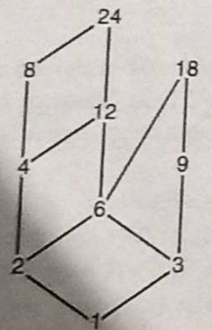
It is transitive relation as xRy, yRz $\forall$x,y, z$\in$N T.P x Rz

$$\left.\begin{array}{l} xRy \Rightarrow \; x \geq y \\ yRz \Rightarrow \; y \geq z \end{array}\right\} \Rightarrow x \geq z \Rightarrow xRz$$

Further the no. of relations from A to B = $2^{O(A)\,O(B)} = 2^{3\times2} = 64$

Q 37. Let A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} be orderd by the relation 'x' divides 'y'. Draw Hasse diagram of this relation. **(PTU, May 2011)**

Solution.



Q 38. Let R be an equivalence relation on a set A. For a, b $\in$A prove that

(a) a $\in$ [a]

(b) b $\in$ [a] if and only if [a] = [b].

(c) two equivalence classes are either identical or disjoint. **(PTU, Dec. 2010)**

Solution. (a) We know that [a] = {x : x $\in$ A, x ~ a} then x $\in$ [a]

as R be equivalence relation on A, if a $\in$ A, and a ~ a $\Rightarrow$ a $\in$ [a] [$\because$ R is reflexive on A]

(b) Given b$\in$ [a] To prove [a] = [b]

Consider any x$\in$ [a] then x ~ a, further b$\in$ [a] $\Rightarrow$ b ~ a as by symmetric property a ~ b also x ~ a, a ~ b by using transitive property x ~ b *i.e.* x$\in$ [b]

Thus [a] $\subset$ [b] ....(1)

Consider any y$\in$ [b] then y ~ b, further b $\in$ [a] then b ~ a, also y ~ b by using transitive property y ~ a *i.e.* y$\in$ [a] Thus [b] $\subset$ [a] ....(2)

From (1) and (2)

[b] $\subset$ [a] and [a] $\subset$ [b] *i.e.* [a] = [b]

Converse [a] = [b] To prove b$\in$ [a]

As, [a] = [b] $\Rightarrow$ b ~ a$\in$ [a] $\Rightarrow$ b$\in$ [a]. [Using part a]

(c) From part (a) and (b), it follows that two different equivalence classes are identical. Further to prove two different equivalence classes are disjoint i.e. if [a] $\neq$ [b] then [a] $\cap$ [b] = ϕ.

Let if possible, [a] $\cap$ [b] $\neq \phi$ i.e. $\exists$ c $\in$A s.t c$\in$ [a] $\cap$ [b] $\Rightarrow$ c$\in$ [a] and c$\in$ [b]

$\Rightarrow$ c ~ a and c ~ b *i.e.* a ~ c, c ~ b [using symmetric property]

$\therefore$ a ~ b using transitive property

$\Rightarrow$ a $\in$ [b] $\Rightarrow$ [a] = [b] which is false.

Thus our supposition is wrong $\therefore$ [a] $\cap$ [b] = ϕ

Hence two equivalence classes are either identical or disjoint.

Q 39. In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi? How many people speak English? **(PTU, Dec. 2011)**

Solution. Let H, E be the set of people who speak Hindi and English respectively

$\therefore$ $n(H) = 35$, $n(H \cap E) = 25$; $n(H \cup E) = 50$

$\therefore$ Required no. of people who speaks English = n (E)

Since $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\therefore$ $n(E) = 50 - 35 + 25 = 40$

$\therefore$ Required no. of people speak only English and not Hindi

$= n(E \cap H^C) = n(E) - n(E \cap H) = 40 - 25 = 15.$

Q 40. Define a relation R defined on Z, the set of all integers as a R b if and only if 7 divides a – b for all a, b $\in$ Z, Show that R is an equivalence relation. **(PTU, Dec. 2011)**

Solution. 1. Reflexivity : Since a – a = 0 = 0.7 $\forall$ a $\in$ Z

$\Rightarrow$ $7 / a - a$ $\Rightarrow$ aRa, thus R is reflexive

2. Symmetric : If aRb $\forall$ a, b $\in$ Z

$\Rightarrow$ $7 / a - b$ $\Rightarrow$ a – b = 7m $\forall$ m $\in$ Z

$\Rightarrow$ b – a = 7 (–m) [$\because$ m $\in$ Z]

$\Rightarrow$ $7 / b - a$ $\Rightarrow$ bRa. Hence, R is symmetric.

3. Transitive : If aRb and bRc $\forall$ a, b, c $\in$ Z

We want to prove that aRc

Now, aRb $\Rightarrow$ $7 / a - b$ $\Rightarrow$ a – b = 7 m ...(1 and $bRc \Rightarrow 7/_{b-c} \Rightarrow b - c = 7n$ .....(2)

on adding (1) and (2) ; we have

$a - c = 7(m + n)$ [as m, n ∈ Z ⇒ m + n ∈ Z]

$\Rightarrow 7/_{a-c} \Rightarrow aRc.$

∴ R is transitive.

Hence, R is reflexive, symmetric and transitive.

∴ R is equivalence relaton on Z.

Q 41. Among the first 1000 positive integers :

(a) Determine the integers which are neither divisible by 5, nor by 7, nor by 9.

(b) Determine the integers divisible by 5 but not by 7, not by 9. (PTU, May 2012)

Solution. Let A, B and C are the set of integers between 1 and 1000 which are divisible by 5, 7 and 9

$$n(A) = n\left(\frac{1000}{5}\right) = 200 \text{ and } n(C) = n\left(\frac{1000}{9}\right) = 111$$

$$n(B) = n\left(\frac{1000}{7}\right) = 142 \text{ and } n(A \cap C) = n\left(\frac{1000}{5.9}\right) = 22$$

$$n(A \cap B) = n\left(\frac{1000}{5.7}\right) = 25 \text{ and } n(B \cap C) = n\left(\frac{1000}{7.9}\right) = 15$$

$$n(A \cap B \cap C) = n\left(\frac{1000}{5.7.9}\right) = 3$$

(a) Required no. of integers $= n(A^C \cap B^C \cap C^C) = n[(A \cup B \cup C)^C]$

$= 1000 - n(A \cup B \cup C)$

$= 1000 - [n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)]$

$= 1000 - [200 + 142 + 111 - 25 - 22 - 15 + 3]$

$= 1000 - 394 = 606$

(b) Required no. of integer $= n(A \cap B^C \cap C^C)$

$= n[A \cap (B \cup C)^C]$

$= n(A) - n(A \cap (B \cup C))$

$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$

$= 200 - 25 - 22 + 3 = 156.$

Q 42. Give an example of a partial order relation on the set ∠1 of integers. (PTU, May 2013 ; Dec. 2012)

Solution. ∀, a, b ∈ I, (a, b) ∈ R ⇒ a ≤ b

Clearly a ≤ a ∀ a ∈ I ∴ (a, a) ∈ R Thus R is reflexive

∀ a, b ∈ I s.t (a, b) ∈ R and (b, a) ∈ R, T.P a = b

Now (a, b) ∈ R ⇒ a ≤ b

(b, a) ∈ R ⇒ b ≤ a ∴ a = b Thus R is antisymmetric

∀ a, b, c ∈ I, (a, b) ∈ R, (b, c) ∈ R, T.P (a, c) ∈ R

Now (a, b) ∈ R ⇒ a ≤ b

(b, c) ∈ R ⇒ b ≤ c ⇒ a ≤ c ⇒ (a, c) ∈ R

∴ R is transitive.

Thus R is reflexive, antisymmetric and transitive ∴ R is partial order relation on I.

□□□

Chapter 3

Ring

QUESTION-ANSWERS

Q 1. What is a ring?

OR (PTU, May 2009, 2007, 2005 ; Dec. 2006, 2003)

Define ring with examples. (PTU, Dec. 2009 ; May 2004)

Solution. It is the algebraic system with two binary operations denoted '+' and '.' resp.

A non empty set R equipped with two binary operations denoted additively '+' and multiplicatively '.' is called a ring and satisfies following axions. ∀ a, bn, c ∈ R

1. a + b ∈ R
2. (a + b) + c = a + (b + c)
3. For any a ∈ R ∃ 0 ∈ R s.t. a + 0 = a = 0 + a
4. For any a ∈ R ∃ b ∈ R s.t. b + a = 0 = a + b
5. a + b = b + a
6. a . b ∈R
7. (a. b). c = a. (b. c)
8. a. (b + c) = a . b + a . c (Distributive law)

A ring is said to be commutative or abeñan ring if ∀ a, b ∈ R, ab = ba.

e.g. I/<4> = {[0], [1], [2], [3]}

The composite table is given as under :

+4	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[2]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

.4	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

Clearly it is a ring under + modulo 4 and ·modulo 4. Also it symmetrical about main diagonal. ∴ It is commutative. So it is a finite commutative ring with unity [1]. Now here we have [2] ≠ [0] ∈ I/<4> but [2]. [2] = [0] (mod 4).

∴ I/<4> has proper zero divisor. ∴ It is not an integral domain.

Q 2. What is a subring? **(PTU, Dec. 2005)**

Solution. A non-empty subset S of ring R is said to be subring of R if it is a ring in itself under borrowed operation of R. {0}, R are the improper subrings of R while any other subring is called proper subring of R.

e.g (Z, +, .) be any ring under the operations of addition and multiplication of integers. Then nZ = {.... –2n, –n, 0, n, 2n} for a proper subring of Z if n ≠ 0, 1, –1.

e.g. Clearly {0, 3} and {0, 2, 4} are proper subring of ring under + modulo 6 and . modulo 6.

Q 3. What is ring without identity? **(PTU, Dec. 2005)**

Solution. It is the algebraic system with two binary operations denoted '+' and '.' resp.

A non empty set R equipped with two binary operations denoted additively '+' and multiplicatively '.' is called a ring and satisfies following axions. ∀ a, bn, c ∈ R

1. a + b ∈ R
2. (a + b) + c = a + (b + c)
3. For any a ∈ R ∃ 0 ∈ R s.t. a + 0 = a = 0 + a
4. For any a ∈ R ∃ b ∈ R s.t. b + a = 0 = a + b
5. a + b = b + a
6. a . b ∈R
7. (a. b). c = a. (b. c)
8. a. (b + c) = a . b + a . c (Distributive law)

A ring is said to be commutative or abelian ring if ∀ a, b ∈ R, ab = ba.

A ring which doesn't contains multiplicative identity *i.e.* 1 is called ring without identity.

For this type of ring we have no x for which x.1 = x = 1.x

e.g. : The set of even integers is a ring without identity.

Q 4. Define Euclidean ring (domain). **(PTU, May 2007, 2006)**

Solution. A commutative ring 'R' is called a Euclidean ring (E.R) if ∀ a ≠ 0 ∈R, we can define a non-negative integer d (a) ≥ 0 called d-value of a and the following conditions are satisfied

(i) d (a) ≤ d (ab) ∀ a, b, ≠ 0 ∈R with ab ≠ 0

(ii) ∀ a, b ≠ 0 in R ∃ q, r ∈ R s.t a = bq + r where either r = 0 or d (r) < d (b).

Q 5. Define improper ideals. **(PTU, Dec. 2007)**

Solution. A non empty subset A of a ring R is called left ideal of R iff (1) A is an additive subgroup of R.

(2) ∀ a ∈ A, r ∈ R ⇒ ra ∈ A

A is said to be right ideal of R iff

(1) A is an additive subgroup of R.

(2) ∀ a ∈ A, r ∈R ⇒ ar ∈ A

Then A is said to be an ideal of R iff (1) A is an additive subgroup (2) ∀ a ∈ A, r ∈ R ⇒ ar, ra ∈ A.

Now, {0} and R are called improper or trivial ideals of R while any other ideal is called proper ideal of R.

Q 6. Define the term, 'an integral domain' and give an example. **(PTU, Dec. 2011 ; May 2009)**

Solution. Let R be a commulative ring is said to be integral domain if it has no proper zero divisor *i.e.* ∀ a, b ∈ R if ab = 0. Then either a = 0 or b = 0 or if a ≠ 0, b ≠ 0 ∈R. Then ab ≠ 0.

e.g : (I, +, .) forms an integral domain.

Q 7. Show that {0} is an ideal in any ring R. **(PTU, May 2010)**

Solution. Let R be any ring

Since ∀ a, –a ∈ R ⇒ a – a ∈R ⇒ 0 ∈ R ⇔ {0} ⊆ R

Further {0} is an additive subgroup of R

also, ∀ a ∈ R, a.0 = 0 = 0.a ∈ {0}

∴ {0} be an ideal in ring. R.

Q 8. What are rings, integral domains and fields? State each of them with an example.

OR **(PTU, Dec. 2003)**

Define the following terms with help of examples

(a) Ring

(b) Fields.

Solution. Ring : It is the algebraic system with two binary operations denoted '+' and '.' resp. **(PTU, Dec. 2009)**

A non empty set R equipped with two binary operations denoted additively '+' and multiplicatively '.' is called a ring and satisfies following axions. ∀ a, b, c ∈ R.

1. a + b ∈ R
2. (a + b) + c = a + (b + c)
3. For any a ∈ R ∃ 0 ∈ R s.t. a + 0 = a = 0 + a
4. For any a ∈ R ∃ b ∈ R s.t. b + a = 0 = a + b
5. a + b = b + a
6. a . b ∈R
7. (a. b). c = a. (b. c)
8. a. (b + c) = a . b + a . c (Distributive law)

A ring is said to be commutative or abelian ring if ∀ a, b ∈ R, ab = ba.

Integral Domain : Let R be a commulative ring is said to be integral domain if it has no proper zero divisor. *i.e.* ∀ a, b ∈ R if ab = 0. Then either a = 0 or b = 0 or if a ≠ 0, b ≠ 0 ∈R. Then ab ≠ 0.

Field : Every commutative ring with unity (1 ≠ 0) in which every non-zero elements are invertible is called field.

e.g. 1. Clearly Z (set of integers) form abelian group under addition also closure, Associative laws and distributive laws holds under multiplication, also commutative laws holds under multiplication, '1' behaves its identity also for set of integers Z. We have ∀ a, b ∈Z, whenever ab = 0 we have a = 0 or b = 0. So it is a commutative infinite ring without zero divisor. So it is an infinite integral domain with unity

e.g. 2. I/<4> = {[0], [1], [2], [3]}

The composite table is given as under :

+4	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[2]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

.4	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

Clearly it is a ring under + modulo 4 and –modulo 4. Also it symmetrical about main diagonal. ∴ It is commutative. So it is a finite commutative ring with unity [1]. Now here we have [2] ≠ [0] ∈ I/<4> but [2]. [2] ≡ [0] (mod 4).

∴ I/<4> has proper zero divisor. ∴ It is not an integral domain.

e.g 3. Let $x = \left(a+\sqrt{2}\,b\right)$, $y = \left(c+\sqrt{2}\,d\right)$ be any two elements of $Q\left(\sqrt{2}\right)$.

Here $x + y = \left(a+\sqrt{2}\,b\right) + \left(c+\sqrt{2}\,d\right) = (a+c) + \left(\sqrt{2}\right)(b+d) \in Q\left(\sqrt{2}\right)$ (as a + c, b + d ∈ Q)

also $xy = \left(a+\sqrt{2}\,b\right)\left(c+\sqrt{2}\,d\right) = [ac + 2bd + \sqrt{2}\,(bc+ad)] \in Q\left(\sqrt{2}\right)$.

∴ $Q\left(\sqrt{2}\right)$ is closed under addition and multiplication.

Further associative laws under addition and multiplication holds in $Q\left(\sqrt{2}\right)$ as these laws holds on rational numbers.

Also $0 = 0 + \sqrt{2}\;0$ behaves as its zero elements of $Q\left(\sqrt{2}\right)$.

Further the additive inverse of any element $a + b\sqrt{2} \in Q\left(\sqrt{2}\right)$ is $-a - b\sqrt{2} \in Q\left(\sqrt{2}\right)$. We can easily verifies the distributive laws.

$\forall\, x, y \in Q\left(\sqrt{2}\right)$ where $x = a + \sqrt{2}\,b$, $y = c + \sqrt{2}\,d$

$$xy = \left(a+\sqrt{2}\,b\right)\left(c+\sqrt{2}\,d\right) = (ac+2bd) + \sqrt{2}\,(bc+ad)$$

$$yx = \left(c+\sqrt{2}\,d\right)\left(a+\sqrt{2}\,b\right) = (ca+2db) + \sqrt{2}\,(cb+da)$$

∴ xy = yx (as commutative law holds on set of rational numbers).

Further $1 = 1 + \sqrt{2}\;.0$ behave as its unity.

Let $x = a + \sqrt{2}\,b$ be any element of $Q\left(\sqrt{2}\right)$ we have $y = \frac{1}{x} = \frac{1}{a+\sqrt{2}\,b}$

i.e. $y = \frac{a-\sqrt{2}\,b}{a^2-2b^2} = \left(\frac{a}{a^2-2b^2}\right) + \sqrt{2}\left(\frac{-b}{a^2-2b^2}\right) \in Q\left(\sqrt{2}\right)$

(Here $a^2 - 2b^2 \neq 0$)

s.t. xy = 1 = yx

∴ Every non-zero element of $Q\left(\sqrt{2}\right)$ have multiplicative inverse. Therefore $Q\left(\sqrt{2}\right)$ form a field.

Q 9. Suppose J is an ideal in a commutative ring R. Show that R/J is commutative.

(PTU, Dec. 2002)

Solution. If J be an ideal of R. Then R/J = {x + J : ∀ x ∈ R} forms a ring under the operation defined by (x + J) + (y + J) = x + y + J and (x + J) (y + J) = xy + J ∀ x + J, y + J ∈ R/J where x, y ∈ R.

Proof : ∀x + J, y + J ∈ R/J where x, y ∈ R

(x + J) + (y + J) = x + y + J ∈ R/J (where x + y ∈ R as x, y ∈ R)

as [(x + J) (y + J)] + (z + J) = (x + y + J) + (z + J) = x + y + z + J

(x + J) { [(y + J) + (z + J)] = (x + J) + (y + z + J) = x + y + z + J

∴ Associative law holds under addition as 0 ∈ R. ∴ 0 + J ∈ R/J ∴ J behave as its zero element for any x ∈ R ⇒ –x ∈ R ⇒ –x + J ∈ R/J

s.t. (x + J) + (–x + J) = [(x + (–x)] + J = 0 + J = J

∴ –x + J be the additive inverse for any element x + J of R/J. ∀ x + J, y + J ∈ R/J. We have (x + J) + (y + J) = x + y + J = (y + J) + (x + J) (as x + y = y + x ∀ x, y ∈ R)

and the multiplicative operation on R/J is defined by (x + J) (y + J) = xy + J where x, y ∈ R.

Now firstly we want to prove that this operation is well defined.

i.e. if x + J = x' + J and y + J = y' + J

Then (x + J) (y + J) = (x' + J) (y' + J) *i.e.* xy + J = x' y' + J

Now $x = x + 0 \in x + J = x' + J \Rightarrow x = x' + a_1,\ a_1 \in J$

$y = y + 0 \in y + J = y' + J \Rightarrow y = y' + a_2,\ a_2 \in J$

∴ $xy + J = (x' + a_1)(y' + a_2) + J = x'y' + x'a_2 + a_1y' + a_1a_2 + J$

[Now $x' \in R,\ a_1 \in J \Rightarrow x'a_1 \in J$, also $a_1 a_2 \in J$,

$y' \in R,\ a_2 \in J \Rightarrow a_1 y' \in J$

$\Rightarrow xy + J = x'y' + J$ [as $x'a_2 + a_1y' + a_1a_2 \in J$ ∴ $x'a_2 + a_1y' + a_1a_2 + J = J$]

The multiplicative operation is clearly closed.

∀ x + J, y + J, z + J ∈ R/J where x, y, z ∈ J

∴ [(x + J) (y + J)] (z + J) = [xy + J] (z + J) = (xy) z + J = x (yz) + J

= (x + J) (yz + J) = (x + J) [(y + J) (z + J)]

Further, $(x + J)[(y + J) + (z + J)] = (x + J)[y + z + J] = x(y + z) + J$
$= xy + xz + J$
$= (xy + J) + (xz + J)$
$= [(x + J)(y + J)] + [(x + J)(z + J)]$

$\therefore$ R/A be a ring under the given two operations.

Given, R is commutative. Now we want to prove that R/J is commutative.

Further, $\forall (x + J), (y + J) \in R/J$ we have $(x + J)(y + J) = xy + J$
$= yx + J$ ($\because$ R is commutative) $= (y + J)(x + J)$

$\therefore$ R/J is commutative.

Q 10. Let D be the ring 2×2 matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Show that D is isomorphic to the complex number C, where D is a field. **(PTU, May 2003)**

Solution. Let us make a mapping from D to C

i.e. $f : D \to C$ defined by $f\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + ib \ \forall \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in D$

Now we shall prove that this mapping, *i.e.* f is homomorphism and one-one onto.

Clearly $\forall\ a + ib \in C \ \exists \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in D$ s.t $f\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + ib$

so f is onto.

$\forall \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in D$

$$f\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right\} = f\begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix} = a + c + i(b + d) = a + ib + c + id$$

$$= f\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + f\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$\therefore$ f is homomorphism

$\forall \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \in D$

s.t. $f\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = f\begin{pmatrix} c & -d \\ d & c \end{pmatrix} \Rightarrow a + ib = c + id$

$\Rightarrow$ $a = c$ and $b = d$ [on comparing real and imaginary parts]

i.e. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ $\therefore$ f is one-one

$\therefore$ f is homomorphism, one-one and onto

$\therefore$ D is isomorphic to C *i.e.* $D \cong C$.

Q 11. What is Quotient ring? Explain with the help of suitable example. **(PTU, Dec. 2007, 2006; May 2005)**

OR

Define a quotient ring and give an example for the same. **(PTU, May 2010, 2009)**

Solution. If A be an ideal of ring R, then

$R/A = \{x + A : \forall x \in R\}$ forms a ring under two operation defined by

$(x + A) + (y + A) = x + y + A$

and $(x + A)(y + A) = xy + A \ \forall x + A, y + A \in R/A$ where $x, y \in R$

is called quotient ring or factor ring.

For example : Let Z be the ring of integers and we will show that ideal generated by 5 *i.e.* $T_5 = \langle 5 \rangle = \{5n ; n \in Z\}$ is an ideal of Z.

$\forall\ x, y \in T_5$ *i.e.* $x = 5n, n \in N$ and $y = 5m, m \in N$

Now $x - y = 5n - 5m = 5(n - m) \in T_5$ [If n, m ∈ N then n – m ∈N]

Also for $r \in R, x \in T_5, rx = r(5n) = 5(rn) \in T_5$ [$\because n \in N, r \in Z \Rightarrow rn \in Z$]

Now $xr = (5n)r = 5(nr) \in T_5$ [as $n \in N, r \in Z \Rightarrow nr \in Z$]

$\therefore T_5$ is an ideal of $Z \Rightarrow Z/T_5$ is a quotient ring

Now,
$T_5 = \{....-10, -5, 0, 5, 10, ...\}$
$T_5 + 1 = \{.....-9, -4, 1, 6, 11, ...\}$
$T_5 + 2 = \{.....-8, -3, 2, 7, 12, ...\}$
$T_5 + 3 = \{.....-7, -2, 3, 8, 13, ...\}$
$T_5 + 4 = \{....-6, -1, 4, 9, 14, ...\}$
$T_5 + 5 = \{..... -5, 0, 5, 10, 15, ...\} = T_5$

$\therefore$ quotient ring $= R/T_5 = \{T_5, T_5 + 1, T_5 + 2, T_5 + 3, T_5 + 4\}$

Q 12. Consider the ring Z_{10} – {0, 1, 2, ...9}
(i) Find the units of Z_{10}.
(ii) Let $f(x) = 2x^2 + 4x + 45$. Find the roots of f (x) over Z_{10}. **(PTU, May 2008)**

Solution. (i) If R be a ring with unity

then $\forall\ a \in R \ \exists\ b \in R$ s.t. $ab = 1 = ba$

then a, b are called units of R.

Now $Z_{10} = \{0, 1, ... 9\}$ form a ring under addition modulo 10 and multiplication modulo 10 with unit '1'.

Now unit of '0' does not exists.

Now $1.1 \equiv 1 \pmod{10}$ $\therefore$ 1 is the unit of Z_{10}.
as $3.7 \equiv 1 \pmod{10}$ $\therefore$ 3, 7 are also units of Z_{10}.
Further $9.9 \equiv 1 \pmod{10}$ $\therefore$ 9 is also units of Z_{10}.

$\therefore$ 1, 3, 7, 9 are units of Z_{10} and the elements 2, 4, 5, 6, 8 have no multiplicative inverse in Z_{10}.

(ii) α be the solution of $f(x) = 0$ in Z_{10} iff $f(\alpha) = 0 \pmod{10}$

Here $f(x) = 2x^2 + 4x + 4$

as $2.1^2 + 4.1 + 4 = 0 \pmod{10}$; $2.2^2 + 4.2 + 4 = 0 \pmod{10}$

$\therefore$ 1, 2 ar the roots of $f(x)$ in Z_{10}.

Further $2.6^2 + 4.6 + 4 = 0 \pmod{10}$; $2.7^2 + 4.7 + 4 = 0 \pmod{10}$

$\therefore$ 6, 7 are also the roots of $f(x)$ in Z_{10}.

Thus 1, 2, 6, 7 are the roots of $f(x)$ over Z_{10}.

Q 13. If R is a ring such that
$a^2 = a \ \forall a \in R$ Prove that
(a) $a + a = 0 \ \forall a \in R$
(b) $a + b = 0 \Rightarrow a = b$
(c) R is commutative ring. **(PTU, May 2009)**

Ans. (a) Given $a^2 = a \ \forall a \in R$

Clearly $\forall$ a, $a \in R$ then $a + a \in R$ ($\because$ closure property holds under addition in R)

$\therefore \quad (a + a)^2 = a + a$

$\Rightarrow \quad (a + a)(a + a) = a + a$

$\Rightarrow \quad a^2 + a^2 + a^2 + a^2 = a + a$

$\Rightarrow \quad a + a + a + a = a + a$

$\Rightarrow \quad a + a = 0$ [Concellation laws holds under addition in ring R]

(b) Given, $a + b = 0$

also, by part (a), $a + a = 0$

$\therefore \quad a + b = a + a$

As concellative laws holds under addition in R.

$\therefore \quad a = b$

(c) $\forall$ a, $b \in R \Rightarrow a + b \in R$

$\therefore \quad (a + b)^2 = a + b$ [$\because a^2 = a \ \forall a \in R$]

$\Rightarrow \quad (a + b)(a + b) = a + b$

$\Rightarrow a(a + b) + b(a + b) = a + b$ [Distributive laws holds]

$\Rightarrow a^2 + ab + ba + b^2 = a + b$

$\Rightarrow \quad a + ab + ba + b = a + b$

$\Rightarrow \quad ab + ba = 0 \Rightarrow ab = ba$ [$\because$ of part (b) if $a + b = 0 \Rightarrow a = b$]

Q 14. Show that every field is an integral domain.
(PTU, Dec. 2011 ; May 2012, 2010)

Solution. Let <F, +, .> be any field $\therefore$ by def. It is a commutative ring. Now we want to prove that F is an integral domain.

For this, let $ab = 0 \ \forall a, b \in F$

Now we want to prove that, $a = 0$ or $b = 0$

Let if us suppose that $a \neq 0 \in F$

$\therefore$ by def. of field every non-zero element of F is invertible $\therefore a^{-1}$ exists.

Now, $ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}.0 \Rightarrow (a^{-1}a)b = a^{-1}.0 \Rightarrow b = 0$.

Thus F is a commutative ring without zero divisors $\therefore$ <F, +, .> is an integral domain.

Q 15. Consider the rings (Z, +, .) and (2Z, +, .) and define $f : Z \to 2Z$ by $f(n) = 2n$ for all $n \in Z$. Is f a ring isomorphism? Justify your answer. **(PTU, Dec. 2010)**

Ans. $\forall$ x, $y \in Z$ we have $\phi(x + y) = 2(x + y) = 2x + 2y = \phi(x) + \phi(y)$ Further $\phi(xy) = 2xy$ but $\phi(x)\phi(y) = (2x)(2y) = 4xy \therefore \phi(xy) \neq \phi(x)\phi(y)$. Hence ϕ is not ring homomorphism.

Q 16. What is Ring Homomorphism? **(PTU, May 2011)**

Ans. Homomoorphism : Let (R, +, .) and (R', +, .) be any two rings. Then $\phi : R \to R'$ is said to be homomorphism iff $f(a + b) = \phi(a) + \phi(b) \ \forall a, b \in R$ and $\phi(ab) = \phi(a)\phi(b) \ \forall a, b \in R$.

If ϕ is 1 − 1 also then ϕ is said to be isomorphism. If ϕ is homo, 1 − 1 and onto. Then R is isomorphic to R' and is denoted by $R \simeq R'$.

If R' = R then homomorphism is called endomorphism. If ϕ is 1 − 1 also then ϕ said to be automorphism.

$\forall$ a, $b \in R$ we have $(ab)i = ab = (ai)(bi)$ also $(a+b)i = a+b = ai + bi \therefore i$ is ring homomorphism.

Q 17. Let M be a ring of 2 × 2 matrices over integers. Consider the set $L = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b \in Z\right\}$. Show that L is a left ideal of M. Is L is right ideal of M?

(PTU, Dec. 2010)

Solution. Given, $L = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} ; a, b \in Z\right\}$

Since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in L \therefore$ L be any non empty subset of M.

Also, $\forall \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \in L$ where a, b, c, $d \in Z$

$\therefore \quad \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} - \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} a-c & 0 \\ b-d & 0 \end{bmatrix} \in L$ [$\forall$ a, b, c, $d \in z$ therefore, $a - c, b - d \in Z$]

$\forall \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \in L$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M$

Then $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} \alpha a + \beta b & 0 \\ \gamma a + \delta b & 0 \end{bmatrix} \in L$ $\left[\begin{array}{l} \because \alpha, \beta, \gamma, \delta, a, b \in Z \\ \therefore \alpha a + \beta b, \gamma a + \delta b \in Z \end{array}\right]$

$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{bmatrix} \notin L$

$\therefore$ L is a left ideal of M but L is not a right ideal of M.

Q 18. Show that the set of real numbers of the form $\{a + b\sqrt{2} : a, b \in Z\}$ is an integral domain. Is it a field? **(PTU, Dec. 2010)**

Ans. Let $x = (a + b\sqrt{2})$, $y = (c + d\sqrt{2})$ be any two elements $Q(\sqrt{2})$

Here $x + y = (a + \sqrt{2}\,b) + (c + \sqrt{2}\,d) = (a + c) + \sqrt{2}\,(b + d) \in Q(\sqrt{2})$ (as a+c, b+d ∈ Q)

Also $xy = (a + \sqrt{2}\,b)(c + \sqrt{2}\,d) = [ac + 2bd + \sqrt{2}\,(bc + ad)] \in Q(\sqrt{2})$

∴ $Q(\sqrt{2})$ is closed under addition and multiplication.

Further associative laws under addition and multiplication holds in $Q(\sqrt{2})$ as these laws holds on rational numbers.

Also $\forall x, y \in Q(\sqrt{2})$ where $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$

∴ $x + y = (a + \sqrt{2}\,b) + (c + \sqrt{2}\,d) = (a + c) + \sqrt{2}\,(b + d)$

$y + x = (c + \sqrt{2}\,d)(a + \sqrt{2}\,b) = (c + a) + \sqrt{2}\,(d + b)$

$x + y = y + x$ ($\because a + c = c + a,\ b + d = d + b$)

∴ Commutative laws under addition holds in $Q(\sqrt{2})$

Also $0 = 0 + 0\sqrt{2}$ behaves as its zero elements of $Q(\sqrt{2})$.

Further the additive inverse of any element $a + b\sqrt{2} \in Q(\sqrt{2})$ is $-a - b\sqrt{2} \in Q(\sqrt{2})$. We can easily verifies the distributive laws.

$\forall x, y \in Q(\sqrt{2})$ where $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$

$xy = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}$

$yx = (c + d\sqrt{2})(a + b\sqrt{2}) = (ca + 2db) + (cb + da)\sqrt{2}$

∴ $xy = yx$ (as commutative law holds on set of rational numbers).

Let $x = a + \sqrt{2}\,b$ be any element of $Q(\sqrt{2})$ we have $y = \frac{1}{x} = \frac{1}{a + \sqrt{2}b}$

i.e. $y = \frac{a - \sqrt{2}b}{a^2 - 2b^2} = \left(\frac{a}{a^2 - 2b^2}\right) + \sqrt{2}\left(\frac{-b}{a^2 - 2b^2}\right) \in Q(\sqrt{2})$

(Here $a^2 - 2b^2 \neq 0$)

s.t. $xy = 1 = yx$

∴ Every non-zero element of $Q(\sqrt{2})$ have multiplicative inverse. Therefore $Q(\sqrt{2})$ form a field. ∴ Obviously it is an integral domain.

Q 19. Consider the set Z together with binary operations ⊕ and ⊗ defined by a ⊕ b = a + b − 1, a ⊗ b = a + b − ab. Show that (Z, ⊕, ⊗) is a ring.

(PTU, Dec. 2010)

Ans. Clearly closure property holds under addition and multiplication ∀ a, b, c ∈ R

$(a \oplus b) \oplus c = (a + b - 1) \oplus c = a + b - 1 + c - 1 = a + b + c - 2$

$a \oplus (b \oplus c) = a \oplus (b + c - 1) = a + b + c - 1 - 1 = a + b + c - 2$

∴ Associative law holds under addition.

Similarly $(a \otimes b) \otimes c = (a + b - ab) \otimes c = a + b - ab + c - (a + b - ab)\,c$

$= a + b + c - ab - ac - bc + abc$

$a \otimes (b \otimes c) = a \otimes (b + c - bc) = a + b + c - bc - a\,(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

∴ Associative laws holds under addition and multiplication in R.

Also $a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$

$a \otimes b = a + b - ab = b + a - ba = b \otimes a$

∴ Commutative law under addition and multiplication holds in R.

$\forall a, b, c \in R$, $a \otimes (b \oplus c) = a \otimes (b + c - 1) = a + b + c - 1 - a\,(b + c - 1)$

$= a + b + c - 1 - ab - ac + a$

and $(a \otimes b) \oplus (a \otimes c) = (a + b - ab) \oplus (a + c - ac)$

$= a + b - ab + a + c - ac - 1$

∴ Distributive laws holds in R.

Further for any $a \in R$ we have $a \oplus 1 = a + 1 - 1 = a = 1 \oplus a$

∴ 1 act as zero element of R.

Further for any $a \in R\ \exists\ b \in R$

s.t. $a \oplus b = 1 \Rightarrow a + b - 1 = 1 \Rightarrow b = 2 - a$

∴ $2 - a$ is the additive inverse of any element $a \in R$.

Let c be the unity of R s.t. $a \otimes c = a \Rightarrow a + c - ac = a$

$\Rightarrow c\,(1 - a) = 0 \Rightarrow c = 0$ is act as its unity element.

Q 20. Define an ideal of a ring and give an example.

(PTU, Dec. 2011)

Solution. Ideal : A non empty subset A of a ring R is called left ideal of R iff (1) A is an additive subgroup of R.

(2) $\forall a \in A, r \in R \Rightarrow ra \in A$

A is said to be right ideal of R iff

(1) A is an additive subgroup of R.

(2) $\forall a \in A, r \in R \Rightarrow ar \in A$

Then A is said to be an ideal of R iff (1) A is an additive subgroup (2) $\forall a \in A, r \in R \Rightarrow ar, ra \in A$.

Since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in A$. ∴ A be non empty subset of R. $\forall \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \in A$.

Then $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ 0 & 0 \end{bmatrix} \in A$ (as $a - c, b - d \in Z$ when $a, b, c, d \in Z$)

$\forall \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in A$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in R$

Then $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & \alpha\beta + b\delta \\ 0 & 0 \end{bmatrix} \in A.$

Further $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a\alpha & b\alpha \\ a\gamma & b\gamma \end{bmatrix} \notin A$

∴ A is not a left ideal of R while it is a right ideal of R.

Q 21. ∀ a, b ∈ G where R is a ring, show that (–a). (–b) = a.b. **(PTU, May 2012)**

Solution. First of all, we shall prove that $a(-b) = -(ab) = (-a)b$

Now, $ab + a(-b) = a[b + (-b)] = a.\,0 = 0$ ∴ by def. of additive inverse of R

We have $a(-b) = -ab$, similarly $(-a)b = -ab$ .....(i)

Now, $(-a)(-b) = -[a(-b)]$ by using (i)

$= -[-(ab)]$ by using (i)

$= ab$

(∵ for additive group $-(-a) = a$).

Q 22. Every field is an integral domain. Give an example to establish that the converse is not true. **(PTU, May 2012)**

Solution. Let F be any field. So F be a commutative ring. Now we want to prove that F is without zero divisors.

For this let $a, b \in F$ s.t. $ab = 0$ .... (1) T.P. either $a = 0$ or $b = 0$ if $a = 0$, matter finishes. Let $a \neq 0$, So every non-zero element is invertible. $\Rightarrow a^{-1} \in F$ ∴ From (1), $a^{-1}(ab) = a^{-1}0$ $\Rightarrow (a^{-1}a)b = 0 \Rightarrow eb = 0 \Rightarrow b = 0$ ∴ F is an integral domain.

Q 23. Give an example of an integral domain which is not a field. **(PTU, Dec. 2013)**

Solution. Now (Z, +, .) forms an infinite integral domain as closure, Associative and distributive laws holds under multiplication, also commutative laws holds under multiplication, 1 behaves as identity (multiplicative) and 0 be the additive identity

also $\forall\, a, b \in z$ s.t $ab = 0 \Rightarrow a = 0$ or $b = 0$

Thus set of integer is without zero divisors.

Thus z the set of integers forms an infinite integral domain but every non-zero element of Z is not invertible as $2 \in Z$ but $\frac{1}{2} \notin Z$ s.t. $2.\frac{1}{2} = 1$ ∴ Z is not a field.

Q 24. Prove that a finite integral domain is a field. **(PTU, Dec. 2013)**

Solution. Let $R = \{0, a_1, a_2, ... a_n\}$ be a finite integral domain and let R' be the set of all non-zero elements of R *i.e.* $R' = [a_1, a_2, ... a_n]$ ∴ R' is also commutative ring without zero divisor. ∴ Cancellation laws holds in R'. Let $a \in R'$, consider the set $S = \{a_1 a, a_2 a .... a_n a\}$ all the members of S are members of R' and no elements of S are zero, further all elements of S are distinct.

Let if possible $a_i a = a_j a$, $i \neq j$. Since $a \neq 0 \Rightarrow a_i = a_j$ a contradiction.

∴ $O(S) = n = O(R')$ and $S \subseteq R'$ ∴ $S = R'$

∴ $a \in R' = S \Rightarrow a = a_i a \Rightarrow a_i$ behaves as left unity for a

Let $a \in R'$, consider the set $T = \{aa_1, aa_2, ... aa_n\}$ all members of T are members of R', No elements of T are zero, further all elements of T are distinct.

∴ $T \subseteq R'$ (∵ $aa_i = aa_j$, $i \neq j$, since $a \neq 0$, $a_i = a_j$, a contradiction)

∴ $O(R') = O(T) = n$

∴ $T = R'$

∴ $\forall\, x \in R' \Rightarrow x \in T \Rightarrow x = aa_j$

∴ $a_i x = a_i(aa_j) = (a_i a)a_j = aa_j = x$ ∴ a_i act as left identity of R' further $a_i 0 = 0$. ∴ a_i act as left identity of R' *i.e.* $ay = y$ $\forall\, y \in R$ and it is denoted by 1.

For the existence of left inverse, $1 \in R' \Rightarrow 1 \in S$

∴ 1 must be some one one of $a_1 a, a_2 a ..... a_n a$

∴ $a_k a = 1$ where a be any non-zero element of R.

∴ a_k be an inverse of a. Hence R be a field.

□□□

Chapter 4

Lattic & Boolean Algebra

QUESTION-ANSWERS

Q 1. Explain the principle of duality. **(PTU, Dec. 2002)**

Solution. Let [B, –, ∧, ∨] be a Boolean Algebra under ≤ and S be a true statement for [B, –, ∧, ∨] if S_1 be obtrained from S by replacing ≤ by ≥ (This is equivalent to turning the graph upside down), ∨ by ∧, ∧ by ∨, 0 by 1 and 1 by 0 then S_1 is also true statement.

Q 2. Let N = {1, 2, 3, ...} be ordered by divisibility, which of the following subset is totally ordered,

(i) (2, 6, 24) **(ii) (3, 5, 15)**

(iii) (2, 9, 16) **(iv) (4, 15, 30)**

Solution. A totally ordered set or toset or a chain is a poset in which every two members are comparable. **(PTU, Dec. 2008)**

Since the set N = {1, 2, ...} under divisibility forms a poset, further 2/6, 2/24, 6/24 ∴ (2, 6, 24) is a toset as every two members are comparable. ∴ Ans (i).

Q 3. State : (i) Absorption law (ii) Idempotent law, in a Boolean algebra.

Solution. **(PTU, May 2010)**

(i) $a \vee a = a$, $a \vee a = a$ (Idempotent Laws)

(ii) $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$ (Absorption Laws)

Q 4. Lattice.

Solution. A poset (P, ≤) is said to form a lattice if ∀ a, b ∈ P. Sup {a, b} and inf {a, b} exists in P. Here we write, **(PTU, May 2009)**

Sup {a, b} = $a \vee b$ (read a join b)

Inf {a, b} = $a \wedge b$ (read a meet b)

other notations like a + b, a. b or $a \cup b$, $a \cap b$ can also be used for sup {a, b} and inf {a, b}.

So in order words lattice is an algebraic system with binary operation ∨ and ∧ and it is denoted by [P ; ∨, ∧].

e.g. 1. Let X be any non empty set and P (X) be the power set of X. then [P (X), ∧, ∨] is a lattice under ⊆.

Since [P (X), ⊆] is a poset.

Here ∀ A, B ∈ P (X), A ∨ B = A ∪ B = union of A and B and A ∧ B = A ∩ B = Intersection of A and B.

Q 5. Minimize the Boolean expression : f = xy ⊕ x'y ⊕ x'y'. (PTU, May 2007, 20

Solution. Let us make the truth table of the following function :

x	y	x'	y'	xy	x'y	x'y'	xy ⊕ x'y	f (x, y)
1	0	0	1	0	0	0	0	0
0	1	1	0	0	1	0	1	1
0	0	1	1	0	0	1	0	1
1	1	0	0	1	0	0	1	1

Where, f (x, y) = xy ⊕ x'y ⊕ x'y'

Now look at those values of f (x, y) when it is equal to 1

Now f (0, 1), f (0, 0), f (1, 1) correspondes to value 1 and corresponding min terms are x'y, x'y', xy

i.e. $f(x, y) = x'y + x'y' + xy = x'(y + y') + xy$

$= x' + xy$ [∵ y + y' = 1

$= (x' + x)(x' + y)$ [a + bc = (a + b) (a + c

$= x' + y$ [∵ x' + x = 1

Q 6. How Boolean Algebra is applicable in Logic circuit? Explain with the help of suitable example. **(PTU, May 2007, 2005 ; Dec. 200**

OR

Discuss various applications of Boolean Algebra in logic circuits. **(PTU, May 2009, 2004 ; Dec. 200**

OR

State De Morgan's laws in Boolean algebra. **(PTU, May 200**

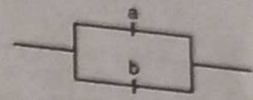
Solution. Boolean Algebra have many useful applications in logic circuits, some of which are given below :

1. Switching Circuits : One of the major applications of Boolean algebra is to the switching systems (an electrical network consisting of switches) that involve two state devices. The simplest example of such a device being an ordinary ON-OFF switch. By a switch we mean a contact or a device in an electric circuit which lets (or does not let) the current to flow through the circuit. The switch can assume two states 'closed' or 'open' (ON or OFF). In the first case the current flows and in the second the current does not flow. We will use a, b, c, ..., x, y, z, ... to denote switches in a circuit. This application is very useful in electrical appliances.

2. Series and parallel connections : There are two basic ways in which switches are generally interconnected. These are referred to as 'in series' and 'in parallel'.

Two switches a, b are said to be connected 'in series' if the current can pass only when both are in closed state and current doesn't flow if any one or both are open. We represent it in the following diagram.

Two switches a, b are said to be connected 'in parallel' if current flows when any one or both are closed and current does not pass when both are open. We represent this by the diagram



If two switches in a circuit be such that both are open (closed) simultaneously, we'll represent them by the same letter. Again if two switches be such that one is open iff the other is closed, we'll represent them by a and a'.

This application is very useful in air condition system and door to the lockers of the bank. In former, we should connect the two switches which are controlled by thermostate in parallel and In latter, we should connect the two switches which are controlled by two keys in series.

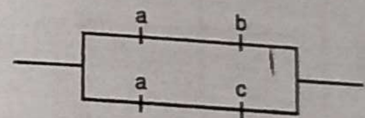
3. Simplification of circuits : Boolean algebra is very useful in simplication of circuits. Simplification of a circuit would normally mean the least complicated circuit with minimum cost and best (convenient) results. This would be governed by various factors like cost of equipment, positioning and number of switches, type of material used etc.

For us, simplification of circuits would mean lesser number of switches, which we achieve by using different properties of Boolean algebra (we may remark here that we are dealing in series-parallel circuits to which Boolean algebra results can be applied. There can, of course, be different (more economical) type of circuits also.)

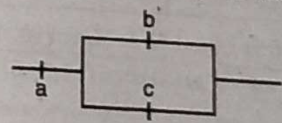
***e.g.* 1.** Let us consider the circuit given by the function

$(a \wedge b) \vee (a \wedge c)$

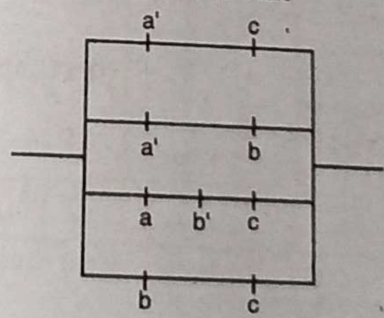
It is represented by



since $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$, this circuit could be simplified to



***e.g.* 2. Consider the circuit**



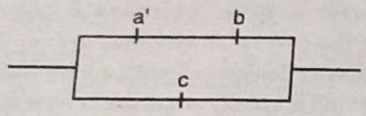
Here, the circuit is represented by the function

$(a' \wedge c) \vee (a' \wedge b) \vee (a \wedge b' \wedge c) \vee (b \wedge c)$

which is equal to

$$(a' \wedge b) \vee (a' \vee (a \wedge b') \vee b) \wedge c$$
$$= (a' \wedge b) \vee [a' \vee (a \wedge b') \vee (a \vee a') \wedge b] \wedge c$$
$$= (a' \wedge b) \vee [a' \vee (a \wedge b') \vee (a \wedge b) \vee (a' \wedge b)] \wedge c$$
$$= (a' \wedge b) \vee [a' \vee \{a \wedge (b' \vee b)\} \vee (a' \wedge b)] \wedge c$$
$$= (a' \wedge b) \vee [a' \vee a \vee (a' \wedge b)] \wedge c$$
$$= (a' \wedge b) \vee [1 \vee (a' \wedge b)] \wedge c = (a' \wedge b) \vee c$$

Thus the circuit is simplified to,



Q 7. Minimize the Boolean expression $f = x'y'z \oplus x'yz' \oplus xyz'$. **(PTU, Dec. 2007)**

Solution. Given function $f = x'y'z \oplus x'yz' \oplus xyz'$

Here we make the truth table of the given function

x	y	z	x'y'z	x'yz'	xyz'	x'y'z ⊕ x'yz'	f (x, y, z)
1	0	0	0	0	0	0	0
0	1	0	0	1	0	1	1
0	0	1	1	0	0	1	1
0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	1
1	0	1	0	0	0	0	0
0	1	1	0	0	0	0	0
1	1	1	0	0	0	0	0

Now look at those values of f (x, y, z) when it is equal to 1. Now f (0, 1, 0) corresponds to min term x'yz' and f (0, 0, 1) corresponds to min term (x'y'z) further f (1, 1, 0) corresponds to min term xyz'

$$\therefore \quad f(x, y, z) = x'yz' + x'y'z + xyz' = x'y'z + (x' + x)\,yz' = x'y'z + yz' \qquad [\because x' + x = 1]$$

Q 8. Using Boolean algebra show that

$abc + ab\bar{c} + a\bar{b}c + \bar{a}bc = ab + ac + bc.$ **(PTU, May 2008)**

Solution. Now, $abc + ab\bar{c} + a\bar{b}c + \bar{a}bc = ab(c + \bar{c}) + a\bar{b}c + \bar{a}bc$

$$= ab + a\bar{b}c + \bar{a}bc \qquad (\because c + \bar{c} = 1)$$

$$= a(b + \bar{b}c) + \bar{a}bc$$

$$= a(b + \bar{b})(b + c) + \bar{a}bc$$
$$= a.1(b + c) + \bar{a}bc = ab + ac + \bar{a}bc \qquad [a + bc = (a + b).(a + c)]$$
$$= ab + c(a + \bar{a}b) = ab + c(a + \bar{a})(a + b)$$
$$= ab + c(a + b) = ab + ac + bc.$$

Q 9. Let (B, +, ., ') is a Boolean algebra. For a ∈ B, if x ∈ B be such that a + x = 1 and a.x = 0, then show that x = a'. Also show that 0' = 1 and 1' = 0. **(PTU, May 2010)**

Solution. Given $a + x = 1$

and $\quad a \cdot x = 0$

Now $\quad a' = a' + 0 = a' + (a.x) = (a' + a).(a' + x)$ ....(1)

i.e. $\quad a' = 1.(a' + x) = a' + x$ ....(2)

Now, $\quad x = x + 0 = x + (a.a') = (x + a).(x + a')$

$\quad = 1.(x + a') = x + a'$

∴ $\quad x = x + a' = a' + x = a'$

Thus $\quad x = a'$

Futher, by $\quad 0 + 1 = 1$ and $0.1 = 0$

by uniqueness of complement, we have

$$0' = 1 \text{ and } 1' = 0.$$

Q 10. Show that the following Boolean expression are equivalent :

(a) $x \wedge (y \vee (y \wedge (y \vee y)))$; $x \wedge y$

(b) $(z \vee x) \wedge ((x \wedge y) \vee z) \wedge (z \vee y)$; $(x \wedge y) \vee z$ **(PTU, Dec. 2014)**

Solution. (a) $x \wedge (y \vee (y \wedge (y \vee y))) = x \wedge (y \vee (y \wedge y))$

$= x \wedge (y \vee y) = x \wedge y$

Now let us draw the truth tables of both expressions

x	y	x∧y	y∨y	y∧(y∨y)	y∨(y(y∨y))	x∧(y∨(y∧(y∨y)))
T	T	T	T	T	T	T
T	F	F	F	F	F	F
F	T	F	T	T	T	F
F	F	F	F	F	F	F

From truth table $x \wedge y \equiv x \wedge (y \vee (y \wedge (y \vee y)))$

(b) $E = (z \vee x) \wedge ((x \wedge y) \vee z) \wedge (z \vee y)$

x	y	z	z∨x	(x∧y)	(x∧y)∨z	z∨y	((x∧y)∨z)∧(z∨y)	E
T	F	F	T	F	F	F	F	F
T	F	T	T	F	T	T	T	T
T	T	F	T	T	T	T	T	T
T	T	T	T	T	T	T	T	T
F	F	F	F	F	F	F	F	F
F	F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F	F
F	T	T	T	F	T	T	T	T

From truth table $(x \wedge y \vee z) \equiv E$

Also

$$E = (z \vee (x \wedge y)) \wedge ((x \wedge y) \vee z) \qquad [\because \text{distributive law holds}]$$
$$= z \vee (x \wedge y) \qquad [\because \text{idempetent law holds}]$$

Q 11. Consider the lattice D_{1001}. The number which divides 1001

(a) Draw the Hasse diagram of D_{1001}.

(b) Find the complement of each number.

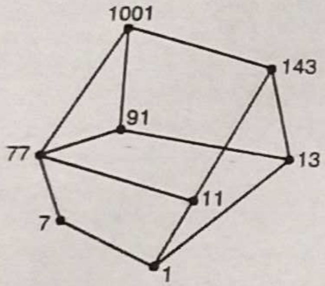
(c) Fine the set A of atoms.

(d) Find the number of subalgebra of D_{1001}. **(PTU, May 2004)**

Solution. (a) D_{1001} = {1, 7, 11, 13, 77, 91, 143, 1001}

It forms Boolean Algebra under relation divisibility. It has least element 1 and greatest element 1001.

Hasse Diagram for D_{1001}



(b) $1 \wedge 1000 = 1$; $1 \vee 1001 = 1001 \Rightarrow \overline{1} = 1001$ & $\overline{1001} = 1$

also $7 \wedge 143 = 1$, $7 \vee 143 = 1001 \Rightarrow \overline{7} = 143$ & $\overline{143} = 7$

also $11 \wedge 91 = 1$, $11 \vee 91 = 1001 \Rightarrow \overline{11} = 91$ & $\overline{91} = 11$

also $13 \wedge 77 = 1$, $13 \vee 77 = 1001 \Rightarrow \overline{13} = 77$ & $\overline{77} = 13$.

(c) Since D_{1001} contains $2^3 = 8$ elements $\therefore$ It has three atoms

(a) $7 \wedge 1 = 1$, $7 \wedge 11 = 1$, $7 \wedge 13 = 1$, $7 \wedge 77 = 7$, $7 \wedge 91 = 7$, $7 \wedge 143 = 1$, $7 \wedge 1001 = 7$

$\therefore 7 \wedge x = 7$ or $1 \; \forall x \in D_{1001}$ $\therefore$ 7 is an atom of D_{1001}

Also, $11 \wedge 1 = 1 = 11 \wedge 7$, $11 \wedge 11 = 11$, $11 \wedge 13 = 1$, $11 \wedge 77 = 11$, $11 \wedge 91 = 1$, $11 \wedge 143 = 11$

$11 \wedge 1001 = 11$ $\therefore$ 11 is an atom of D_{1001}.

Also, $13 \wedge 1 = 1$, $13 \wedge 7 = 1$, $13 \wedge 11 = 1$, $13 \wedge 13 = 13$, $13 \wedge 77 = 1$, $13 \wedge 91 = 13$, $13 \wedge 143 = 13$

$13 \wedge 1001 = 13$ $\therefore$ 13 is an atom of D_{1001}

$\therefore D_{1001}$ has 3 atoms i.e. 7, 11, 13.

(d) A subalgebra of D_{1001} has two, four, eight elements because it has eight elements

(i) D_{1001} has one two element subalgebra which consists of least element '1' and greatest element 1001 i.e. {1, 1001}

(ii) The only eight element subalgebra is D_{1001} itself.

(iii) Any four element subalgebra is of the form {1, x, x', 1001} i.e. which contains least and greatest element and non bound element and its complement. Now there are six such non bound elements in D_{1001}. So there are $\frac{6}{2} = 3$ pairs {x, x'}. So D_{1001} has three four element subalgebras.

In total D_{1001} has 1 + 1 + 3 = 5 subalgebras.

Q 12. State various postulates of Boolean Algebra. State and prove at least five theorems of Boolean Algebra. **(PTU, Dec. 2006, 2004)**

Solution. A complemented, distributive lattice is called boolean algebra. **OR**

Boolean algebra is a system consisting of non-empty set L together with two binary operations $\wedge$ and $\vee$ and a unary operation '–' satisfying $\forall$ a, b, c $\in$ L

(i) $a \wedge a = a = a \vee a$ (idempotency)

(ii) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ (commutativity)

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$ (Associativity)

(iv) $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$ (Associative)

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \vee c)$ (Distributive law)

(vi) $\forall a \in L \; \exists \; \bar{a} \in L$ s.t. $a \wedge \bar{a} = 0 = 0$, $a \vee \bar{a} = 1$

Where 0, 1 are elements of L satisfying $0 \le x \le 1 \; \forall x \in L$ and it is denoted by [B, –, $\wedge$, $\vee$].

e.g. 1. Let A be any set then [B, –, $\wedge$, $\vee$], B = P (A) is said to form Boolean algebra under $\subset$.

Sol. Since [B, $\wedge$, $\vee$] is said to form Lattice under $\subset$. Here X, Y $\in$ B = P (A), $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$ also $X \cap (Y \cap Z) = (X \cap Y) \cup (X \cap Z)$ (Distributive law holds in set theory)

$\therefore$ [B, $\wedge$, $\vee$] is form Distributive lattice also it has least element ϕ and greatest element 'A' and complement of every element of B is exist. $\therefore$ [B, –, $\wedge$, $\vee$] is a complemented and distributive lattice.

$\therefore$ [B, –, $\wedge$, $\vee$] is said to form Boolean Algebra.

e.g. 2. Let B = {0, a, b, 1} Define $\wedge$, $\vee$ and complemented '–' by

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

–	
0	1
a	b
b	a
1	0

Then [B, –, $\wedge$, $\vee$] is said to form Boolean Algebra under these operations.

Sol. Clearly [B, $\wedge$, $\vee$] is a lattice also it has least element 0 and greatest element 1 and complement of every element is exist. $\therefore$ It is complemented lattice.

Also $0 \wedge (a \vee 1) = 0 \wedge 1 = 0$ ..... (1) $0 \wedge (a \vee 1) = (0 \wedge a) \vee (0 \vee 1)$ (Using (1) and (2))

& $(0 \wedge a) \vee (0 \vee 1) = 0 \vee 0 = 0$ .... (2)

again taken any other three elements a, b, 1

$a \wedge (b \vee 1) = a \wedge 1 = a$, $(a \wedge b) \vee (a \vee 1) = 0 \vee a = a$

$\therefore$ It is a Distributive lattice.

Hence [B, –, $\wedge$, $\vee$] is forms Boolean Algebra.

Basic Laws of Boolean Algebra :

(i) $a \vee a = a$, $a \vee a = a$ (Idempotent Laws)

(ii) $a \vee b = b \vee a$, $a \vee b = b \vee a$ (Commutative laws)

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$; $a \vee (b \wedge c) = (a \wedge b) \wedge c$ (Associative Laws)

(iv) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$; $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (Distributive Laws)

(v) $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$ (Absorption Laws)

(vi) $a \vee 0 = a = 0 \vee a$; $a \wedge 1 = a = 1 \wedge a$ (Identity Laws)

(vii) $a \vee \bar{a} = 1$, $a \wedge \bar{a} = 0$ (Complement Laws)

(viii) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$ and $\overline{a \wedge b} = \bar{a} \vee \bar{b}$ (De-Morgan's laws)

(ix) $\left(\bar{\bar{a}}\right) = a$ [Involution laws]

(x) $a \vee 1 = 1$; $a \wedge = 0 = 0$ (Null laws)

Some proof are given as under :

I. Null Laws : $a \vee 1 = 1$; $a \wedge 0 = 0$ or $a + 1 = 1$; $a.\ 0 = 0$

Proof : (i) $a \vee 1 = (a \vee 1) \wedge 1 [\because (a \vee 1) \wedge 1 = (a \wedge 1) \vee (1 \wedge 1) = a \vee 1]$

$$= (a \vee 1) \wedge (a \vee \bar{a})$$

$$= a \vee (1 \wedge \bar{a}) = a \vee \bar{a} = 1$$

(ii) $a \wedge 0 = a \wedge (a \wedge \bar{a}) = (a \wedge a) \wedge \bar{a} = a \wedge \bar{a} = 0.$

II. Demorgan's Laws :

(i) $\overline{a \wedge b} = \bar{a} \wedge \bar{b}$ (ii) $\overline{a \vee b} = \bar{a} \vee \bar{b}$ or $\overline{a + b} = \bar{a}.\bar{b}$

Or

$$\overline{ab} = \bar{a} + \bar{b}$$

Proof : (i) $(a \wedge b) \wedge \bar{a} \vee \bar{b} = \left[(a \wedge b) \wedge \bar{a}\right] \vee \left[(a \wedge b) \wedge \bar{b}\right]$

$$= \left[(a \wedge \bar{a}) \wedge b\right] \vee \left[a \wedge \left(b \wedge \bar{b}\right)\right]$$

$$= [0 \wedge b] \vee [a \wedge 0] = 0 \vee 0 = 0$$

Now $(a \wedge b) \vee (\bar{a} \vee \bar{b}) = \left[\left(\bar{a} \vee \bar{b}\right) \vee a\right] \wedge \left[\left(\bar{a} \vee \bar{b}\right) \vee b\right]$

$$= \left[(\bar{a} \vee a) \vee \bar{b}\right] \vee \left[\bar{a} \vee \left(\bar{b} \vee b\right)\right]$$

$$= \left[1 \vee \bar{b}\right] \wedge \left[\bar{a} \vee 1\right] = 1 \wedge 1 = 1$$

Hence $(\overline{a \wedge b}) = \bar{a} \vee \bar{b}$

(ii) $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = \left[\left(\bar{a} \wedge \bar{b}\right) \wedge a\right] \vee \left[\left(\bar{a} \wedge \bar{b}\right) \wedge b\right]$

$$= \left[(\bar{a} \wedge a) \wedge \bar{b}\right] \vee \left[\bar{a} \wedge \left(\bar{b} \wedge b\right)\right]$$

$$= \left[0 \wedge \bar{b}\right] \vee \left[\bar{a} \wedge 0\right] = 0 \vee 0 = 0$$

Now $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = \left[(a \vee b) \vee \bar{a}\right] \wedge \left[(a \vee b) \vee b\right]$

$$= \left[(a \vee \bar{a}) \vee b\right] \wedge \left[a \vee \left(b \vee \bar{b}\right)\right]$$

$$= (1 \vee b) \wedge [a \wedge 1] = 1 \wedge 1 = 1$$

III. Involution Laws : $\left(\bar{\bar{a}}\right) = a$

Now $a \wedge \bar{a} = 0$; $a \vee \bar{a} = 1$

Also, $\bar{a} \wedge \bar{\bar{a}} = 0$; $\bar{a} \vee \bar{\bar{a}} = 1$

$\Rightarrow 0 = a \wedge \bar{a} = \bar{\bar{a}} \wedge \bar{a}$; $a \vee \bar{a} = \bar{\bar{a}} \vee \bar{a} = 1$

$\Rightarrow a = \bar{\bar{a}}.$

IV. (i) If $a \le b$ iff $\bar{a} \ge \bar{b}$ (ii) $a \le b \Leftrightarrow a \wedge b' = 0$

Sol. (i) $a \le b \Rightarrow a \wedge b = a \Rightarrow \bar{a} = \left(\overline{a \wedge b}\right) = \bar{a} \vee \bar{b} \Rightarrow \bar{b} \le \bar{a} \Rightarrow \bar{a} \ge \bar{b}$

Converse : $\bar{a} \ge \bar{b} \Rightarrow \bar{\bar{a}} \le \bar{\bar{b}}$ (using part (i)) $\Rightarrow a \le b$

(ii) $a \le b \Rightarrow a \wedge b' \le b \wedge b' = 0$ also $0 \le a \wedge b' \le 0 \Rightarrow a \wedge b' = 0.$

V. In boolean algebra B, $\bar{0} = 1$, $\bar{1} = 0$

Since $0 + 1 = 1$ and $0.1 = 0$

$\therefore$ by uniqueness of complement, $\bar{1} = 0$ and $\bar{0} = 1.$

Q 13. State that commutative laws, associative laws and absorption law for lattices. **(PTU, Dec. 2005)**

Solution. (i) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ (Commutativity)

$$a \wedge b = \text{Inf}\{a, b\} = \text{Inf}\{b, a\} = b \wedge a$$

$$a \vee b = \text{Sup}\{a, b\} = \text{Sup}\{b, a\} = b \vee a$$

(ii) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (Associativity)

Let $b \wedge c = d \Rightarrow d = \text{Inf}\{b, c\} \Rightarrow d \le b, d \le c$ ....(1)

Let $e = \text{Inf}\{a, d\} \Rightarrow e \le a, e \le d$ ....(2)

$\therefore$ from (1) and (2) $e \le a, e \le b, e \le c$

$\therefore$ e be the lower bound of a, b, c.

Le f be any other lower bound of a, b, c

$f \le a, f \le b, f \le c$ but $d = \text{Inf}\{b, c\}$

$\therefore f \le d, f \le a$ but $e = \text{Inf}\{a, d\} \Rightarrow f \le e$

$\therefore e = \text{Inf}\{a, b, c\} = a \wedge d = a \wedge (b \wedge c)$

Similarly $(a \wedge b) \wedge c = \text{Inf}\{a, b, c\}$

$\therefore \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c.$

(iii) $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$ (Absorption)

Now a ≤ b, a ≤ a ⇒ a be the lower bound of a and b

⇒ a ≤ a ∧ b but by def. of a ∧ b we have a ∧ b ≤ a

∴ by antisymmetric property of lattice a ∧ b = a

Converse : a ∧ b = a T.P. a ≤ b

a = a ∧ b = Inf. {a, b} ⇒ a ≤ a, a ≤ b. Hence proved. Now, similarly we can prove the second part.

Now a ≤ a ∨ b (using vii)

∴ a ∧ (a ∨ b) = a (by using v)

Q 14. Simplify f algebraically where f $(x_1, x_2, x_3) = \overline{(x_1 + x_2)\, x_3} \cdot (x_1 + x_2)$. Also express the result graphically.

(PTU, Dec. 2006)

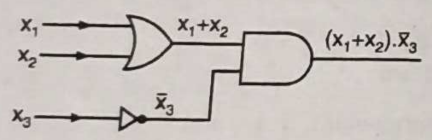
Solution. $f(x_1, x_2, x_3) = \overline{((x_1 + x_1)\, x_3)} \cdot (x_1 + x_2)$

$$= \left(\overline{x_1 + x_2} + \bar{x}_3\right) \cdot (x_1 + x_2)$$

$$= \left[\left(\overline{x_1} + x_1\right) \cdot (x_1 + x_2)\right] + \bar{x}_1 \cdot (x_1 + x_2)$$

$$= \bar{x}_3 \cdot (x_1 + x_2) = \bar{x}_3 x_1 + \bar{x}_3 x_2$$

Gate diagram for result is as follows :



Q 15. Minimize the following switching function : Σm (0, 2, 10, 11, 12, 14).

(PTU, May 2007, 2006)

Solution. Here the greatest min term is 14 i.e. N = 14

Also for the number of variables for the K-map is given $N \le 2^n$ which is only satisfies when n = 4 i.e. we form four variable K-map.

Here we use A, B variables along horizontal and C, D variables along vertical lines and min terms are expressed as '1'in the table.

CD \ AB	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

CD \ AB	00	01	11	10
00	1			1
01				
11	1			1
10			1	1

Here the best possible quadrat and one pair as shown in table and the minimizing form of the switching function is given by $\overline{B} + AC\overline{D}$.

Q 16. Minimize the following switching function Σm (1, 5, 6, 7, 11, 12, 13, 15).

(PTU, Dec. 2007)

Solution. Here the greatest min term is 15 i.e. N = 15 also for the number of variables for K-map is given by $N \le 2^n$ it is satisfies if n = 4 i.e. we form four variable K-map.

(i) Here we use A, B variable along horizontal and C, D variables along vertical lines and min terms can be expresses as '1' in the table.

CD \ AB	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

CD \ AB	00	01	11	10
00		1		
01		1	1	1
11	1	1	1	
10			1	

In K-map square min terms forms 1 quadrants and 4 pairs of '1' s.

∴ The minimizing form is given as under

$BD + A\overline{C}D + \overline{A}B\overline{C} + \overline{A}CD + ABC$.

Q 17. Let L be a bounded distributive lattice. Show that if a complement exists it is unique.

(PTU, Dec. 2010, 2008)

Solution. Let a ∈ L and it has two distinct complements a', a'' (say)

∴ a ∧ a' = 0 and a ∨ a' = 1

a ∧ a'' = 0 and a ∨ a'' = 1

Now a' = a' ∧ 1 = a' ∧ (a ∨ a'')

= (a' ∧ a) ∨ (a' ∧ a'')

= 0 ∨ (a' ∧ a'')

= a' ∧ a'' ....(1)

a'' = a'' ∧ 1 = a'' ∧ (a ∨ a')

= (a'' ∧ a) ∨ (a'' ∧ a')

= 0 ∨ (a'' ∧ a')

= a'' ∧ a' (commutativity)

= a' ∧ a'' ....(2)

∴ From (1) and (2)

a' = a'' which gives a contradiction to our supposition i.e. a has two distinct complement.

∴ complement of every element of L is unique.

e.g. Find the complement of each element of Lattice D_{42}.

Sol. D_{42} = {1, 2, 3, 6, 7, 14, 21, 42}

It has least element I and greatest element 42.

Q 18. Simplify using Boolean postulates and theorems
$a + ab + abc + abcd + \bar{a} + \bar{a}b + \bar{a}bcd$ **(PTU, May 200...)**

Solution. $a + ab + abc + abcd + \bar{a} + \bar{a}b + \bar{a}bcd$

$= a(1+b) + abc + abcd + \bar{a}(1+b) + \bar{a}bc + \bar{a}bcd$

$= a.1 + abc + abcd + \bar{a}.1 + \bar{a}bc + \bar{a}bcd$

$= a + abc(1+d) + \bar{a} + \bar{a}bc(1+d)$ $[\because 1 + b = 1]$

$= a + abc.1 + \bar{a} + \bar{a}bc.1$

$= a + \bar{a} + (a + \bar{a})bc$

$= (a + \bar{a}) + 1.bc$ $(\because a + \bar{a} = 1)$

$= 1 + 1.bc = (1+b)(1+c)$ $[\because 1 + bc = (1+b).0+0]$

$= 1.1 = 1.$

Q 19. In any Boolean algebra B, prove that,
$(a' \vee b') \vee (a \wedge b \wedge c') = (b \wedge c') \vee (a' \vee b')$. **(PTU, Dec. 2011)**

Solution. L.H.S $= (a' \vee b') \vee (a \wedge b \wedge c')$

$= [(a' \vee b') \vee a] \wedge [(a' \vee b') \wedge (b \wedge c')]$ [Distributive law holds]

$= [(a' \vee a) \vee (b' \vee a)] \wedge [(a' \vee b') \wedge (b \wedge c')]$

$= [1 \vee (b' \vee a)] \wedge [(a' \vee b') \wedge (b \wedge c')]$ [Complement laws holds]

$= 1 \wedge [(a' \vee b') \wedge (b \wedge c')]$ [Identity laws]

$= (a' \vee b') \wedge (b \wedge c')$ $[\because 1 \wedge a = a]$

$= (b \wedge c') \wedge (a' \vee b')$ $[\because$ Commutative laws holds $[a \vee b = b \vee a$ and $a \wedge b = b \wedge a]$

$=$ R.H.S.

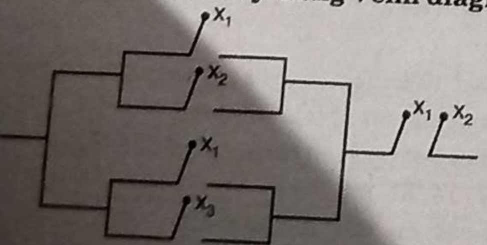
Q 20. Poset. **(PTU, May 2009)**

Solution. First of all we define partial order relation, a Relation R on set A is said to be partial order relation iff

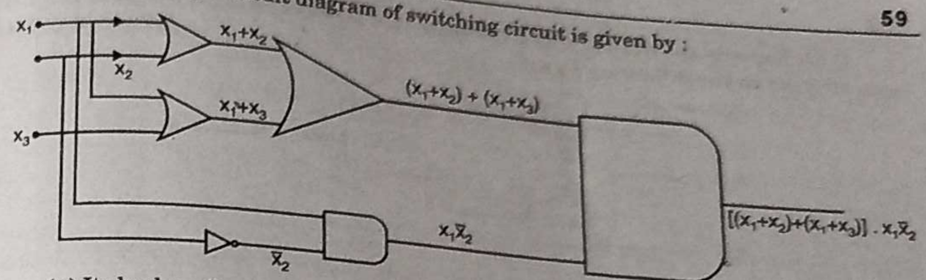
(i) R is reflexive
(ii) R is antisymmetric
(iii) R is transitive

Then the set A with partial order Relation R is said to be poset.
e.g. $(P[A], \subseteq)$ forms a Poset.
Where P(A) = power set of A.

Q 21. Express the switching circuit shown in the figure through the logic or gate circuit.
(a) Write Boolean function.
(b) Simplify the function f algebraically.
(c) Find the minterm normal form by using Venn diagram and express it in gate diagram. **(PTU, Dec. 2010)**



Solution. Logic circuit diagram of switching circuit is given by :



(a) Its boolean function be

$f(x_1, x_2, x_3) = [(x_1 + x_2) + (x_1 + x_3)].\, x_1\bar{x}_2$

(b) $f(x_1, x_2, x_3) = [x_1 + x_2 + x_3].\, x_1\bar{x}_2$ $[\because a + a = a]$

$= x_1.(x_1\bar{x}_2) + x_2.(x_1\bar{x}_2) + x_3.(x_1\bar{x}_2)$

$= x_1\bar{x}_2 + 0 + x_3.\, x_1\bar{x}_2$ $\begin{bmatrix} a.a = a \\ a.\bar{a} = 0 \end{bmatrix}$

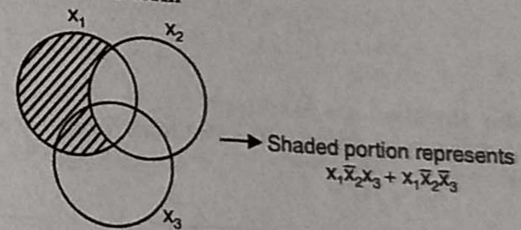
$= x_1\bar{x}_2 + x_3\, x_1\bar{x}_2$

$= (1 + x_3)\, x_1\bar{x}_2$

$= x_1\bar{x}_2$

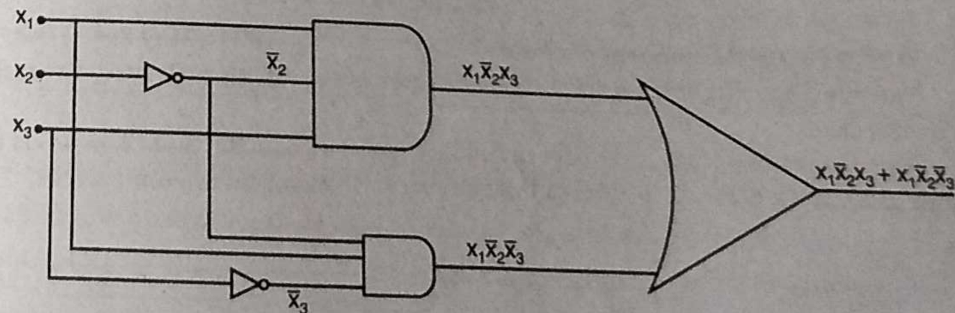
(c) $f(x_1, x_2, x_3) = x_1\bar{x}_2\,(x_3 + \bar{x}_3) = x_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3$ $[\because 1 + a = 1]$

This gives the minterm normal form ....(1)

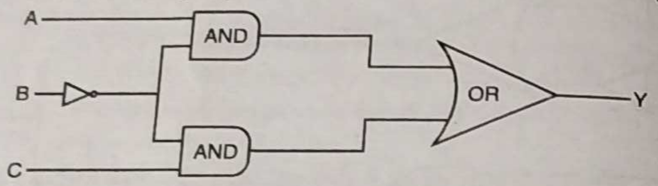


shaded portion represents $x_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3$

Logic circuit for f (x, y, z) is given as under



Q 22. Express the output Y as a Boolean expression in the inputs A, B, C for the logic circuits in the following figure (PTU, May 2011)



Solution. $Y = A\overline{B} + \overline{B}C$.

Q 23. Write the dual of each of the following Boolean equations :
(a) (x + 0) (1.x) = 1, (b) x + x'y = x + y (PTU, Dec. 2011)

Solution. (a) $(x + 0)(1.x) = 1 \Rightarrow x.x = 1$ [using identity laws]

$\Rightarrow$ $x = 1$ [using idempotent laws]

The dual is given by $x = 0$

(b) Given, $x + x'y = x + y$

$\Rightarrow$ $(x + x').(x + y) = x + y$ [using Distributive laws]

$\Rightarrow$ $1.(x + y) = x + y$ [using compliment laws]

$\Rightarrow$ $x + y = x + y$ [Identity laws]

The dual is given by

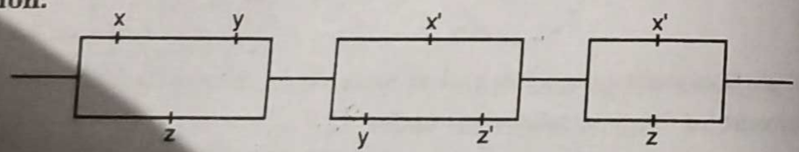
$x. y = x. y.$

Q 24. In a Boolean algebra B, show that, a + a = a ∀ a ∈ B. (PTU, May 2012)

Solution.
$a = a + 0$ [using identity laws]
$= a + a.a'$ [using compliment laws]
$= (a + a).(a + a')$ [using distributive laws]
$= (a + a).1$ [using complement laws]
$= (a + a)$ [using identity laws]

Q 25. Consider the Boolean function,s f (x, y, z) = (x. y + z). (x' + y.z') . (x' + z) construct the circuit corresponding to the Boolean function of the Boolean algebra of switching circuits. (PTU, May 2012)

Solution.



Q 26. Explain the concept of chain. (PTU, May 2007 ; Dec. 2006)

Solution. A totally ordered set or toset or a chain is a poset in which every two members are comparable.

Since the set N = {1, 2, ...} under divisibility forms a poset, further (2, 6, 24) is totally ordered as 2/6, 2/24, 6/24 ∴ (2, 6, 24) is a toset as every two members are comparable.

❑❑❑

Chapter 5

Permutation & Combination

QUESTION-ANSWERS

Q 1. Define permutation with examples. (PTU, May 2007, 2004)

Solution. By permutation, we mean to arrange some or all of given things and we denote it by ^nP_r.

For example : We can arrange three letters a, b, c taking two at a time as follows: ab, bc, ca, ba, cb, ac.

i.e. $^nP_r = \dfrac{n!}{(n-r)!}$ *i.e.* number of permutations of n different things taking r at a time.

Q 2. Define combination with examples. (PTU, May 2007, 2004)

Solution. It is the selection of some or all things at a time out of a given number of things.

For example: If we have to select a class representative from a number of students in a class or selecting eleven players out of twenty etc.

So, total number of combinations of n distinct things taking r at a time is denoted by the following symbols–

$$^nc_r \text{ or } c(n, r) \text{ or } \binom{n}{r}$$

Where $1 \le r \le n$.

Q 3. How many 4-digit telephone number have one or more repeated digits? (PTU, May 2006 ; Dec. 2004)

Solution. For 4-digit telephone number, unit, tenth, hundred and thousand place can be filled with 10 ways each.

∴ Total number of 4-digit telephone numbers = 10 × 10 × 10 × 10 = 10000

Now, total number of 4-digit telephone number when no digit repeated = 10 × 9 × 8 × 7
= 5040

∴ Required number of 4-digit telephone number have one or more repeated digits
= 10000 – 5040 = 4960

Q 4. What is a circular permutation of n objects and how many are there? **(PTU, Dec. 2007)**

Solution. Since, at a round table, there is no first, no last position. So, let us fix the position of one person.

∴ remaining (n - 1) persons can be arranged at n – 1 Places in $^{n-1}P_{n-1} = (n-1)!$

The circular permutationof n things at a round table = (n – 1)!

Which includes the total number of permutations considering clockwise & anti-clockwise directions both.

In these two arrangements, each person has the same neighbours though in the reverse order & one can be obtained from the other.

$$\therefore \text{number of distinct permutations} = \frac{1}{2}(n-1)!$$

[this method is applicable in case of beads]

Q 5. In how many ways can a president and vice president be chosen from a set of 30 candidates?

(i) 820 (ii) 850

(iii) 880 (iv) 870 **(PTU, May 2008)**

Solution. The required number of ways in which a president and vice president can be chosen from 30 candidates

$$= {}^{30}P_2 = \frac{30!}{28!} = 30 \times 29 = 870 \text{ ways} \therefore \text{Ans. (iv)}$$

Q 6. Consider four vowels and eight consonants. Find the number m of five letter words containing two different vowels and three different consonants that can be formed from the given letters. **(PTU, Dec. 2003)**

Solution. Given there are four vowels and eight consonents.

2 vowels can be selected out of 4 in $^4C_2$ ways and 3 consonents can be choosen out of 8 in $^8C_3$ ways.

Then by fundamental principle of counting, we have

$$\text{Required number of ways} = {}^4C_2 \times {}^8C_3 = \frac{4!}{2!2!} \times \frac{8!}{5!3!}$$

$$= 6 \times 56 = 336 \text{ ways.}$$

Q 7. Suppose that three are n-people in a room, n ≥ 1 and that they all shake hands with one another. Prove that $\frac{n(n-1)}{2}$ hand shakes will have occurred. **(PTU, Dec. 2005)**

Solution. Given, total number of persons in a room = n

To shake hands, we require two hands of two different persons,

$$\therefore \text{total no. of handshakes} = {}^nC_2 = \frac{n(n-1)}{2}.$$

Q 8. What is the basic principle of counting? Explain. **(PTU, Dec. 2006, 2005, 2003 ; May 2006)**

Solution. Sum and product rules : There are mainly two types of fundamental principles of counting.

(i) Fundamental principle of mutliplication : If there are two events such that one of them can be completed in m ways and when it has been completed in any one of the m ways, second event can be completed in n ways then the two events in succession can be completed in mn ways.

(ii) Fundamental principle of addition : If there are two events such that they can be performed indepenently in m and n ways respectively then either of the two events can be performed in (m + n) ways.

e.g 1. Let us suppose that there are 10 male professors and 5 female professors to teach mathematics subject in an enginnering colleges and we want to know that, in how many ways a student can choose a mathematics professor.

The answer to this problem is given by

Sum Rule principle i.e. 10 + 5 = 15 ways.

e.g 2. How many 5 letter words can be formed from the word 'KNIFE' if repitition of letters is not allowed.

Total number of letters used in the word KNIFE = 5

Five-letters words can be formed in the following way -

Ist place can be filled up in 5 ways

2nd place can be filled up in 4 ways

3rd place can be filled up in 3 ways (∵ repetition of letters is not allowed)

4th place can be filled up in 2 ways

5th place can be filled up in 1 way

∴ total number of 5-letter words = 5×4×3×2×1= 120

Q 9. How many words can be obtained by arranging the letters of the word 'UNIVERSAL' in different way? In how many of them

(a) E, R, S occur together

(b) No two of the letters E, R, S occur together. **(PTU, May 2008)**

Solution. Number of letters in word UNIVERSAL = 9

∴ Required number of words = 9! = 362880

(a) Here let us suppose the letters E, R, S consider as single letter ∴ total no. of letters will be 1 + 6.

(Where one letter as ERS and remaining letters of word are U, N, I, V, A, L)

and this can be permute in 7! ways

Further the letters E, R, S can be permute in 3! ways.

∴ Required number of ways = 3! × 7! = 30240 ways.

(b) When the letters E, R occur together then total no. of letters will be (1 + 7)

(Where one letter ER and remaining letters of word universal is U, N, I, V, S, A, L) and this can be permute in 8! ways and E, R can be permute 2! ways.

$\therefore$ Total no. of words in which E, R occur togethers = $2! \times 8!$

Similarly when E, S and R, S occur together, total no. of words = $8! \times 2!$ (for each)

Thus required no. of words = $9! - 3 \times 2! \times 8! = 8! \times 3$ ways.

Q 10. A set contains (2n + 1) elements. If the number of subsets of this set which contain at most n elements is 8192. Find the value of n. **(PTU, Dec. 2008)**

Solution. Total no. of elements in a set = 2n + 1

$\therefore$ Total no. of subsets of this set having 0 element = ${}^{2n+1}C_0$

Total no. of subsets of this set having 1 element = ${}^{2n+1}C_1$

and so on.

Total no. of subsets having n elements = ${}^{2n+1}C_n$

$\therefore$ Total no. of subsets of this set having atmost n-elements

$$= {}^{2n+1}C_0 + {}^{2n+1}C_1 \;.....\; + {}^{2n+1}C_n$$

We know that, ${}^{2n+1}C_0 + {}^{2n+1}C_1 \;.....\; + {}^{2n+1}C_{2n+1} = 2^{2n+1}$ ...(1)

Also, we know that, ${}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r}$

$\therefore$ (1) becomes ; ${}^{2n+1}C_0 + \;.....\; + {}^{2n+1}C_n = 2^{2n}$

According to given condition, we have

${}^{2n+1}C_0 + \;.....\; + {}^{2n+1}C_n = 8192$

i.e. $2^{2n} = 8192 = 2^{13}$

$\Rightarrow$ $2n = 13$

$\Rightarrow$ $n = 6.5.$

Q 11. Show that the following statements are equivalent :

P_1 : n is even integer.

P_2 : n – 1 is an odd integer.

P_3 : n^2 is an even integer. **(PTU, Dec. 2008)**

Solution. First of all we prove that $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3$ and $P_3 \Rightarrow P_1$

Given P_1 : n is even integer $\therefore$ n = 2K where $K \in I$

$\Rightarrow n - 1 = 2K - 1$ = odd integer $\therefore P_1 \Rightarrow P_2$ ...(1)

Let P_2 : n – 1 is an odd integer

$\Rightarrow$ $n - 1 = 2K - 1$ where $K \in I$

$\Rightarrow$ $n = 2K \Rightarrow n^2 = 4K^2$ = even integer

thus $P_2 \Rightarrow P_3$ ...(2)

Let P_3 : n^2 is an even integer.

$\Rightarrow$ $n^2 = 4K^2$ where $K \in I$

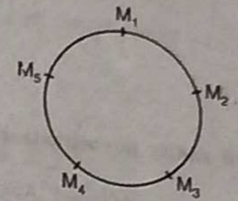
$\Rightarrow$ $n = \pm 2K$ $\therefore$ n is an even integer $\therefore P_3 \Rightarrow P_1$

From (1), (2) and (3), we have all three statements are equivalent.

Q 12. In how many ways can 5 Gentle man and 5 ladies be seated round a table so that no two ladies are together. **(PTU, May 2008)**

Solution. Clearly, 5 gentle man be seated in round table in (5–1)! ways

i.e. 4! ways. Clearly no two ladies are together for such arrangement, we have five vacant places for 5 ladies.



This can be done is 5! ways.

$\therefore$ Total no. of arrangement = $4! \times 5! = 24 \times 120 = 2880$ ways.

Q 13. Write short notes on the following :
Sum and product rules. **(PTU, May 2007 ; Dec. 2006)**

Solution. Sum and product rules : There are mainly two types of fundamental principles of counting.

(i) Fundamental principle of mutliplication : If there are two events such that one of them can be completed in m ways and when it has been completed in any one of the m ways, second event can be completed in n ways then the two events in succession can be completed in mn ways.

(ii) Fundamental principle of addition : If there are two events such that they can be performed indepenently in m and n ways respectively then either of the two events can be performed in (m + n) ways.

Q 14. Prove that ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$. **(PTU, May 2011 ; Dec. 2010)**

Ans.

$$\text{R.H.S.} = {}^nc_r + {}^nc_{r-1} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!} = \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\,\frac{n-r+1+r}{r(n-r+1)}$$

$$= \frac{n!}{(r-1)!(n-r)!}\,\frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} \qquad \left\{\begin{array}{l}\because (n+1)! = (n+1)n! \\ (n-r+1)(n-r)! = (n-r+1)!\end{array}\right.$$

$$= \text{L.H.S.} \qquad \left(\because {}^{n+1}C_r = \frac{(n+1)!}{r!(n+1-r)!}\right)$$

Q 15. Find 'n' if P (n, 2) = 72. (PTU, May 2011)

Solution. Given $P(n, 2) = 72 \Rightarrow {}^nP_2 = 72$

$$\Rightarrow \frac{n!}{(n-2)!} = 72 \Rightarrow n(n-1) = 72 \Rightarrow n^2 - n - 72 = 0$$

$$\Rightarrow (n-9)(n+8) = 0 \Rightarrow n = 9, -8$$

$\therefore n = 9$ as $n = -8$ is impossible.

Q 16. Define permutation. How many permutations are possible on a set S = (1, 2, 3, 4, 5). (PTU, May 2011)

Solution. Permutation : Let A be a non empty set. Any 1-1 onto mapping $f : A \to A$ is called a permutation.

$\therefore$ Total no. of permutations = 5! = 120.

Q 17. Find the product of the following permutations

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix}.$$ (PTU, May 2011)

Solution. $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{pmatrix}.$

Q 18. A bag contains six white marbles and five red marbles. Find the number of ways four marbles can be drawn from the bag if

(a) they can be any color

(b) two must be white and two red. (PTU, May 2011)

Solution. (a) Required no. of ways = $\underset{\left(\begin{array}{c}\text{all four}\\ \text{are white}\end{array}\right)}{\frac{{}^6C_4}{{}^{11}C_4}} + \underset{\left(\begin{array}{c}\text{all four}\\ \text{ar}\end{array}\right)}{\frac{{}^5C_4}{{}^{11}C_4}}$

$$= \frac{(6\times5\times4\times3)}{11\times10\times9\times8} + \frac{5\times4\times3\times2}{11\times10\times9\times8} = \frac{2}{23}$$

(b) Required no. of ways = $\frac{{}^6C_2 \times {}^5C_2}{{}^{11}C_4} = \frac{6\times5\times5\times4\times4\times3\times2}{11\times10\times9\times8\times2\times2} = \frac{5}{11}.$

Q 19. In how many ways can nine students be partitioned into three teams containing four, three and two students respectively? (PTU, Dec. 2013)

Solution. Required no. ways = $\frac{9!}{3!\,4!\,3!\,2!} = \frac{9\times8\times7\times6\times5}{6\times3\times2\times1\times2} = \frac{1260}{6} = 210$ ways.

❑❑❑

Chapter 6

Recurrence Relation & Logics

QUESTION-ANSWERS

Q 1. Define the recurrence relation. (PTU, Dec. 2002)

Solution. For a numeric function (S(0), S(1), ...S(r),.....) an equation relating S(r) for any r to one or more S (i) where i < r is called a recurrence relation.

A recurrence relation of the type

$S(k) + c_1 S(k-1) \ldots\ldots + c_n S(k-n) = f(k)$

Where c_1 ....., c_n are constants, is called linear recurrence relation

eg : $2S(k) + 4S(k-1) + 7S(k-2) = 2^k$

Q 2. What is meant by tautology? (PTU, Dec. 2003)

Solution. Tautology and Contradiction : Tautology is a statement which has truth value T for all possible values of the statement. A contradiction is a statement which has truth value F for all possible values of the statement.

e.g : P∨~P is tautology, while P∧~P is contradiction and this can be cleared from their truth tables.

P	~P	P∨~P
T	F	T
F	T	T

all values are T.

It is a tautology

P	~P	P∧~P
T	F	F
F	T	F

For all values are F.

It is contradiction.

Q 3. Construct truth table for the following formula : $\sim (P \vee \sim Q) \Leftrightarrow (P \Rightarrow Q)$. **(PTU, Dec. 20..)**

Solution. The truth table is given as under for the expression $\sim (P \vee - Q) \Leftrightarrow (P \Rightarrow Q)$

P	Q	$\sim Q$	$P \vee \sim Q$	$\sim (P \vee \sim Q)$	$P \Rightarrow Q$	$\sim (P \vee \sim Q) \Leftrightarrow (P \Rightarrow Q)$
T	F	T	T	F	F	T
F	T	F	F	T	T	T
F	F	T	T	F	T	F
T	T	F	T	F	T	F

Q 4. Discuss recurrence relations. **(PTU, May 2007 ; Dec. 20..)**

Solution. Linear recurrence Relation with Constant Coefficients

A recurrence relation of the type

$S(k) + c_1 S(k-1) \ldots\ldots + c_n S(k-n) = f(k)$

Where $c_1 \ldots\ldots, c_n$ are Constants, is called linear recurrence relation

eg : $2 S(k) + 4S(k-1) + 7 S(k-2) = 2^k$

Fibonacci Sequence

The sequence starts with two numbers 1,1 and having numbers which are the sum of its two immediate predecessors.

Mathematically, $a_0, a_1 = 1, a_k = a_{k-1} + a_{k-2}, k \geq 2$ or $S(0) = 1, S(K), S(1) = S(k-1) + S(k-2)$

Homegeneous Recurrence Relation :

A linear Recurrence Relation is given by

$S(k) + c_1 S(k-1) \ldots\ldots + c_n S(k-n) = f(k)$

If $f(k) = 0$ then the Recurrence relation is called Homegeneous recurrence relation. Otherwise it is said to be non-homogeneous recurrence relation.

Q 5. The statement $(p \wedge q) \Rightarrow p$ is a:

(i) Contingency **(ii) Absurdity**

(iii) Tautology **(iv) None of the above** **(PTU, May 200..)**

Solution.

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	F	F	T
F	T	F	T
T	T	T	T
F	F	F	T

$\therefore$ It is a tautology thus Ans. (iii)

Q 6. $p \rightarrow q$ is logically equivalent to

(i) $\sim q \rightarrow p$ **(ii) $\sim p \rightarrow q$**

(iii) $\sim p \wedge q$ **(iv) $\sim p \vee q$** **(PTU, Dec. 200..)**

Solution. Here we use truth table method

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	F	F	F	F
F	T	T	T	T
T	T	T	F	T
F	F	T	T	T

$\therefore$ Truth table values for $p \rightarrow q$ and $\sim p \vee q$ are same.

Q 7. What is the generating function for the sequence $S_n = 2^n$? **(PTU, May 2010)**

Solution. Given $S_n = 2^n$

$$\therefore \quad G(S, Z) = \sum_{n=0}^{\infty} 2^n Z^n = \sum_{n=0}^{\infty} (2Z)^n = 1 + \sum_{n=1}^{\infty} (2Z)^n$$

$$= 1 + \sum_{n+1=1}^{\infty} (2Z)^{n+1} = 1 + 2Z \sum_{n=0}^{\infty} (2Z)^n$$

$$= 1 + 2Z\, G(S, Z)$$

$$\Rightarrow \quad G(S, Z) = \frac{1}{1-2Z}$$

Q 8. Solve the following recurrence equations using the techniques for linear recurrence relations with constant coefficients : $a_n - 6a_{n-1} + 8a_{n-2} = 0$ and $a_0 = 1, a_1 = 10$. **(PTU, May 2004)**

Solution. The given recurrence equation is

$a_n - 6a_{n-1} + 8a_{n-2} = 0$

Let $a_n = a^n$ be its solution so eq (1) ....(1)

having characteristic equation is given as under

$a^n - 6a^{n-1} + 8a^{n-2} = 0$

$\Rightarrow a^{n-2}(a^2 - 6a + 8) = 0 \Rightarrow a^2 - 6a + 8 = 0$

$\Rightarrow \quad a = 2, 4$

$\therefore$ General solution is given by

$a_n = C_1 2^n + C_2 4^n$

Now, $a_0 = 1 \Rightarrow 1 = C_1 + C_2$ ....(2)

and $a_1 = 10 \Rightarrow 10 = 2C_1 + 4C_2$ ....(3)

on solving (3) and (4), we have ....(4)

$C_2 = 4 ; C_1 = -3$

$\therefore$ Complete solution is given by $a_n = -3.2^n + 4.4^n$

i.e. $a_n = -3.2^n + 4^{n+1}$.

Q 9. Find the generating function for the Fibonacci sequence. **(PTU, May 2005, 2003)**

Solution. Here, the recurrence relation is given by

$S(k) - S(k-1) - S(k-2) = 0, k \geq 2$ with $S(0) = 1, S(1) = 1$

Multiply given eq. by z^k and summing from 2 to ∞, we have

$$\sum_{k=2}^{\infty} S(k)z^k - \sum_{k=2}^{\infty} S(k-1)z^k - \sum_{k=2}^{\infty} S(k-2)z^k = 0 \qquad ....(1)$$

$$\sum_{k=2}^{\infty} S(k)z^k = \left[S(0)+S(1)z+\sum_{k=2}^{\infty} S(k)z^k - S(0)-S(1)z\right]$$

$$= G(S;z)-1-z \qquad ....(2)$$

$$\sum_{k=2}^{\infty} S(k-1)z^k = z\sum_{k=2}^{\infty} S(k-1)z^{k-1} = z\sum_{k=1}^{\infty} S(k)z^k = z\left[S(0)+\sum_{k=1}^{\infty} S(k)z^k - S(0)\right]$$

$$= z\left[\sum_{k=0}^{\infty} S(k)z^k - S(0)\right] = z\,[G(S;z)-1] \qquad ....(3)$$

$$\sum_{k=2}^{\infty} S(k-2)z^k = z^2\sum_{k=2}^{\infty} S(k-2)z^{k-2} = z^2\sum_{k=0}^{\infty} S(k)z^k = z^2G(S;z) \qquad ....(4)$$

putting eq. (2), (3), (4) in eq (1), we get

$[G(S;z)-1-z] - z\,[G(S;z)-1] - z^2\,G(S;z) = 0$

$\Rightarrow \quad G(S;z)\,[1-z-z^2] = 1$

$$\Rightarrow \quad G(S;z) = \frac{1}{\left(1-\frac{1+\sqrt{5}}{2}z\right)\left(1-\frac{1-\sqrt{5}}{2}z\right)}$$

$$= \frac{A}{1-\frac{1+\sqrt{5}}{2}z} + \frac{B}{1-\frac{1-\sqrt{5}}{2}z}$$

where $A = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}$ and $B = \frac{-1}{\sqrt{5}}\frac{1-\sqrt{5}}{2}$.

$$\therefore \quad S(k) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}$$

Q 10. Solve the recurrence relation $a_r - 2a_{r-1} + a_{r-2} = 0$ given that $a_0 = 1$ and $a_1 = 2$. **(PTU, May 2006)**

Solution. Let the homongeneous solution be $a_r = a^r$ by substituting in the given recurrence relation

We get the characteristic equation is

$a^r - 2a^{r-1} + a^{r-2} = 0$

$\Rightarrow a^{r-2}(a^2 - 2a + 1) = 0$

$\Rightarrow \quad a = 1, 1$

$\therefore \quad a_r = (C_1 + C_2 r)\,1^r = C_1 + C_2 r \qquad ....(1)$

Now $\quad a_0 = 1 \Rightarrow 1 = C_1$

and $\quad a_1 = 2 \Rightarrow 2 = C_1 + C_2 \Rightarrow C_2 = 1$

$\therefore$ equations (1) gives ;

$a_r = 1 + r.$

Q 11. Discuss an algorithm of solving nth order linear homogeneous recurrence relation. **(PTU, Dec. 2006)**

Solution. Homegeneous Recurrence Relation :

A linear Recurrence Relation is given by

$S(k) + c_1S(k-1) ...... + c_nS(k-n) = f(k)$

If $f(k) = 0$ then the Recurrence relation is called Homegeneous recurrence relation.

Characteristic Equation :

Let the Homogenous relation be $S(k)+c_1S(k-1) ......+ c_n S(k-n) = 0$

Then its characteristic eq. is nth degree polynomial equation.

$$a^n + \sum_{i=1}^{n} c_i\, a^{n-i} = 0$$

i.e. $a^n + c_1 a^{n-1} + c_2 a^{n-2} ...... + c_n = 0$

Note : We put in general $S(k) = a^k$ in given relation and taking the least power of a^k common from given relation, the rest becomes its characteristic equation.

Algorithm to Solve Homogeneous Recurrence Relation

(i) Find the characteristic equation of given relation.

(ii) Solve the characteristic eq and hence find its Roots called characteristic roots

Case-I. If the roots are distinct say $a_1, a_2, a_3 ......, a_n$

Then $\quad S(k) = B_1a_1k + B_2a_2k + ... + B_na_nk$

Where B_1, ... Bn are arbitrary constants.

Case-II. If m roots of n roots are equal say $a_1 = a_2 ...... = a_m, a_{m+1}, ... a_n$

Then $\quad S(k) = (B_1 + B_2k + ... + B_mk_{m-1})\, a_1k + B_{m+1}\, a_{m+1}k + ... + B_na_nk$

Solution of Non-Homogeneous Recurrence Relation :

A Recurrence Relation of the type

$S(k) + c_1 S(k-1) + c_2 S(k-2) ..... + c_nS(k-n) = f(k) \qquad ....($

Where $\quad f(k) \neq 0$

is called Non–Homogeneous Recurrence Relation.

Its solution i.e particular solution depends upon the function f (k).

Case-I. When $f(k)$ is constant

Let the particular solution be $S(k) = d$

$\therefore$ eq (1) gives, $d + c_1 d + c_2 d \ldots\ldots + c_n d = f(k)$

$$d = \frac{f(k)}{1 + c_1 \ldots\ldots + c_n} \quad \text{if } 1 + c_1 \ldots\ldots + c_n \neq 0$$

Now if $\quad 1 + c_1 \ldots\ldots + c_n = 0$, procedure fails

Then we take $S(k) = kd$ If this procedure also fails then $S(k) = k_2 d$ and so on.

Case-II. When $f(k)$ is a linear function $= q_0 + q_1 k$

Then particular solution be $S(k) = A_0 + A_1 k$

Case-III. When $f(k) = q_0 + q_1 k \ldots\ldots + q_m k_m$

Then particular Solution be $S(k) = A_0 + A_1 k + \ldots\ldots + A_m k_m$

Case-IV. When $f(k)$ = exponential function $= q\, a^k$

Then the particular Solution be $S(k) = A\, a_k$

Note : If the particular solution contain any term of homogeneous solution then we multiply the particular solution by k and try this solution and repeat the process unless the particular solution does not contain any term of homogeneous solution.

Q 12. Solve the recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$, given that $a_0 = 0, a_1 = 3$.
(PTU, Dec. 2007)

Solution. Let $S(k) = a^k$ by substituting, the characteristic equation given by,

$$a^k - 7a^{k-1} + 10a^{k-2} = 0$$

$$\Rightarrow \quad a^{k-2}(a^2 - 7a + 10) = 0$$

$$\Rightarrow \quad a^2 - 7a + 10 = 0 \Rightarrow a = 2, 5 \text{ are the characteristic roots}$$

$\therefore$ general solution $S(k) = c_1 2^k + c_2 5^k$ ......(1)

$S(0) = 0 \Rightarrow 0 = c_1 + c_2$ ......(2)

$S(1) = 3 \Rightarrow 3 = 2c_1 + 5c_2$ ......(3)

Solving (2) & (3), we get

$$c_2 = 1, \; c_1 = -1$$

$\therefore$ eq (1) gives, we get

$$S(k) = -2^k + 5^k \text{ is the required solution.}$$

Q 13. Solve the recurrence relation $a_n = -3a_{n-1} + 10a_{n-2}$, $n \geq 2$, given $a_0 = 1, a_1 = -4$.
(PTU, Dec. 2009)

Solution. The given recurrence relation be

$a_n - 3a_{n-1} - 10a_{n-2} = 0$ .....(1)

putting $\quad a_n = a^n$ in eq (1) ;

we have the characteristic equation is

$$a^n + 3a^{n-1} - 10a^{n-2} = 0$$

$$\Rightarrow a^{n-2}(a^2 + 3a - 10) = 0$$

i.e. $a^2 + 3a - 10 = 0 \Rightarrow (a-2)(a+5) = 0$

$$\Rightarrow \quad a = 2, -5$$

$\therefore$ General solution is given by

$a_n = C_1 2^n + C_2 (-5)^n$ ...(2)

Given, $\quad a_0 = 1 \Rightarrow 1 = C_1 2^0 + C_2(-5)^0$ i.e. $1 = C_1 + C_2$ ...(3)

also, $\quad a_1 = -4 \Rightarrow -4 = 2C_1 - 5C_2$ ...(4)

on solving (3) and (4) ; we get

$$C_2 = \frac{6}{7} \; ; C_1 = \frac{1}{7}$$

$\therefore$ From (2) ; we have

$$a_n = \frac{1}{7} 2^n + \frac{6}{7}(-5)^n = \frac{1}{7}[2^n + 6(-5)^n] \text{ is the required solution.}$$

Q 14. Solve the recurrence relation $S(n) - 6S(n-1) + 9S(n-2) = 3^{n+1}$.
(PTU, May 2010)

Solution. The associated homogeneous recurrence relation be

$S(n) - 6S(n-1) + 9S(n-2) = 0$ ...(1)

Its characteristic equation can be found out

by putting $\quad S(n) = a^n$ in eq (1),

i.e. $a^n - 6a^{n-1} + 9a^{n-2} = 0$

$\Rightarrow a^{n-2}(a^2 - 6a + 9) = 0 \Rightarrow a = 3, 3$

$\therefore \quad S_h(n) = (C_1 + C_2 n)\, 3^n$

Here $f(n) = 3^{n+1}$

Let its particular solution be $S(n) = d.3^{n+1}$ but this contains a term which is present in its homogeneous solution so let us try $S(n) = dn.3^{n+1}$ as particular solution but this also contains a term which is present in homogeneous solution. So let us try $S(n) = n^2.d.3^{n+1}$ as particular solution.

$\therefore$ eq (1) becomes ;

$n^2 d\, 3^{n+1} - 6(n-1)^2 d\, 3^n + 9(n-2)^2 d\, 3^{n-1} = 3^{n+1}$

$\Rightarrow d\, 3^{n-1}[9n^2 - 18(n-1)^2 + 9(n-2)^2] = 3^{n+1}$

$\Rightarrow d\,[n^2 - 2(n-1)^2 + (n-2)^2] = 1$

$$\Rightarrow \quad d[-2 + 4] = 1 \Rightarrow d = \frac{1}{2}$$

$$\therefore \quad S_p(n) = \frac{1}{2} n^2 . 3^{n+1}$$

$$\therefore \quad C.S = S(n) = S_h(n) + S_p(n) = (C_1 + C_2 n)\, 3^n + \frac{1}{2} n^2 . 3^{n+1}$$

i.e. $$C.S = S(n) = (C_1 + C_2 n + \frac{3}{2} n^2)\, 3^n.$$

Q 15. Solve the following recurrence relation :
$S(K) - 10S(K-1) + 9S(K-2) = 0$. Where $S(0) = 3$ $S(1) = 11$. **(PTU, May 20**

Solution. The given recurrence relation is

$S(K) - 10S(K-1) + 9S(K-2) = 0$

Let $\quad S(K) = a^K$ be the solution of eq (1)

$\therefore$ Its characteristic equation is given by

$a^K - 10\,a^{K-1} + 9a^{K-2} = 0$

$\Rightarrow a^{K-2}(a^2 - 10a + 9) = 0$

$\Rightarrow a^2 - 10a + 9 = 0 \Rightarrow a = 1, 9$

$\therefore \quad S(K) = C_1 \cdot 1^K + C_2\, 9^K$ ....(2)

Now $\quad S(0) = 3 \Rightarrow 3 = C_1 + C_2$ ....(3)

and $\quad S(1) = 11 \Rightarrow 11 = C_1 + 9C_2$ ....(4)

Solving (3) and (4) ; we get

$C_2 = 1 \,;\, C_1 = 2$

$\therefore$ eq (2) gives ; $S(K) = 2 + 9^K$.

Q 16. Solve the recurrence relation $a_{r+2} - 3a_{r+1} + 2a_r = 0$, by the method of generating function with the initial conditions $a_0 = 2$ and $a_1 = 3$. (PTU, Dec. 2014)

Solution. Given recurrence relation is $a_{r+2} - 3a_{r+1} + 2a_r = 0$,

$a_0 = 2$ and $a_1 = 3$ ....(1)

Multiply eqn (1) by z^{r+2} and summing from 0 to ∞

$$\therefore \sum_{r=0}^{\infty} a_{r+2}\, z^{r+2} - 3\sum_{r=0}^{\infty} a_{r+1}\, z^{r+2} + 2\sum_{r=0}^{\infty} a_r\, z^{r+2} = 0 \quad ....(2)$$

$$\text{Now, } \sum_{r=0}^{\infty} a_{r+2}\, z^{r+2} = \left[a_0 + a_1 z + \sum_{r=0}^{\infty} a_{r+2}\, z^{r+2} - a_0 - a_1 z\right]$$

$$= \sum_{r=0}^{\infty} a_r z^r - 2 - 3z = G(z) - 2 - 3z \quad ....(3)$$

$$\sum_{r=0}^{\infty} a_{r+1}\, z^{r+2} = z\sum_{r=0}^{\infty} a_{r+1}\, z^{r+1} = z\left[a_0 + \sum_{r=0}^{\infty} a_{r+1}\, z^{r+1} - a_0\right] = z\left[\sum_{r=0}^{\infty} a_r z^r - 2\right]$$

$$= z\,[G(z) - 2] \quad ....(4)$$

$$\& \quad \sum_{r=0}^{\infty} a_r\, z^{r+2} = z^2 \sum_{r=0}^{\infty} a_r\, z^r = z^2\, G(z)$$

Using (3), (4) & (5) in eqn (2) ; we get

$G(z) - 2 - 3z - 3z\,[G(z) - 2] + 2z^2 G(z) = 0$

$\Rightarrow G(z)\,[1 - 3z + 2z^2] = -3z + 2$

$$\Rightarrow \quad G(z) = \frac{2-3z}{2z^2 - 3z + 1} = \frac{2-3z}{(1-z)(1-2z)} = \left[\frac{1}{1-z} + \frac{1}{1-2z}\right]$$

$$\therefore \quad a_r = 1(1)^r + 1\,(2)^r = 1^r + 2^r \,;\, r \geq 0$$

Q 17. Solve the recurrence relation S (K) + 5S (k – 1) + 6S (k – 2) = f (k),

$$\textbf{Where } f(k) = \begin{cases} 0, k = 0, 1, 5 \\ = 6, \text{ otherwise} \end{cases}$$ **, given that S (0) = S (1) = 2. (PTU, Dec. 2006)**

Solution. The homogeneous relation is

$S(K) + 5S(K-1) + 6S(K-2) = 0$ ....(1)

Let $\quad S(K) = a^K$ by substituting in eq (1), we have

$a^K + 5a^{K-1} + 6a^{K-2} = 0 \Rightarrow a^{K-2}(a^2 + 5a + 6) = 0$

$\Rightarrow \quad a = -2, -3$

$\therefore$ The homogeneous solution is given by

$S_h(K) = a_1(-2)^K + a_2(-3)^K$

For the particular solution, we consider two cases.

Case-I. When $f(K) = 0$ i.e. when $K = 0, 1, 5 \therefore S_p(K) = 0$

Then $\quad S(K) = a_1(-2)^K + a_2(-3)^K$ ....(2)

also $\quad S(0) = 2 \therefore$ eq (3) gives ; $2 = a_1 + a_2$ ....(3)

and $\quad S(1) = 2 \therefore$ eq (3) gives ; $2 = -2a_1 - 3a_2$ ....(4)

on solving (4) and (5), we get ....(5)

$-6 = a_2$ and $a_1 = 8 \therefore S(K) = 8(-2)^K - 6(-3)^K$

Case-II. When $K \neq 0, 1, 5$, $f(K) = 6$

Let us try $S(K) = d$ for particular solution

$\therefore$ given recurrence relation becomes

$$d + 5d + 6d = 6 \Rightarrow d = \frac{1}{2} \therefore S_p(K) = \frac{1}{2}$$

$$\therefore S(K) = S_h(K) + S_p(K) = a_1(-2)^K + a_2(-3)^K + \frac{1}{2} \quad .....(6)$$

$$\text{Given,} \quad S(0) = 2 \Rightarrow 2 = a_1 + a_2 + \frac{1}{2} \Rightarrow a_1 + a_2 = \frac{3}{2} \quad ....(7)$$

$$\text{and} \quad S(1) = 2 \Rightarrow 2 = -2a_1 - 3a_2 + \frac{1}{2} \Rightarrow -2a_1 - 3a_2 = \frac{3}{2} \quad ....(8)$$

on solving (7) and (8) ; we get

$$-a_2 = \frac{9}{2} \Rightarrow a_2 = -\frac{9}{2} \text{ and } a_1 = 6$$

Thus eq (6) gives ;

$$S(K) = 6(-2)^K - \frac{9}{2}(-3)^K + \frac{1}{2}$$

Q 18. Show that

$\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q) \Leftrightarrow (\neg P \vee Q)$

Solution. (PTU, May 2008)

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
P	Q	~P	~(P∨Q)	~P∨Q	~P∨(~P∨Q)	~(P∧Q)→ ~P∨(~P∨Q)
T	F	F	T	F	F	F
T	T	F	F	T	T	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T

From (v) & (vii), we have

(~P∨Q)⇔ ~(P∧Q)→ (~P∨ (~P∨Q))

Q 19. Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$, and $a_1 = 12$. **(PTU, May 2010)**

Solution. The given recurrence relation can be written as

$$a_k - a_{k-1} - 2a_{k-2} = 2^k \text{ ; where } a_0 = 4 \text{ and } a_1 = 12 \quad ....(1)$$

Multiply the given eq (1) by Z^k and sum up from 2 to ∞.

i.e. $$\sum_{k=2}^{\infty} a_k Z^k - \sum_{k=2}^{\infty} a_{k-1} Z^k - 2\sum_{k=2}^{\infty} a_{k-2} Z^k = \sum_{k=2}^{\infty} 2^k Z^k \quad ....(2)$$

Now, $$\sum_{k=2}^{\infty} a_k Z^k = \left[a_0 + a_1 Z + \sum_{k=2}^{\infty} a_k Z^k - a_0 - a_1 Z\right] = \sum_{k=0}^{\infty} a_k Z^k - 4 - 12Z$$

$$= [G(a, Z) - 4 - 12Z] \quad ....(3)$$

$$\sum_{k=2}^{\infty} a_{k-1} Z^k = Z\sum_{k=2}^{\infty} a_{k-1} Z^{k-1} = Z\sum_{k=1}^{\infty} a_k Z^k = Z[G(a, Z) - 4] \quad ....(4)$$

and $$\sum_{k=2}^{\infty} a_{k-2} Z^k = Z^2 \sum_{k=2}^{\infty} a_{k-2} Z^{k-2} = Z^2 \sum_{k=0}^{\infty} a_k Z^k = Z^2 G(a, Z) \quad ....(5)$$

and $$\sum_{k=2}^{\infty} (2Z)^k = 1 + 2Z + \sum_{k=2}^{\infty} (2Z)^k - 1 - 2Z$$

$$= \sum_{k=0}^{\infty} (2Z)^k - 1 - 2Z = \frac{1}{1-2Z} - 1 - 2Z = \frac{4Z^2}{1-2Z} \quad ....(6)$$

Putting (3), (4), (5) and (6) in eq (2) ; we have

$$[G(a, Z) - 4 - 12Z] - Z[G(a, Z) - 4] - 2Z^2 G(a, Z) = \frac{4Z^2}{1-2Z}$$

$$G(a, Z)[1 - Z - 2Z^2] = 4 + 8Z + \frac{4Z^2}{1-2Z}$$

$$\Rightarrow -G(a, Z)[2Z^2 + Z - 1] = \frac{4(1-4Z^2) + 4Z^2}{1-2Z} = \frac{4(1-3Z^2)}{1-2Z}$$

$$\Rightarrow \quad G(a, Z) = \frac{-4(1-3Z^2)}{(1-2Z)(Z+1)(2Z-1)} = \frac{4(1-3Z^2)}{(2Z-1)^2 (Z+1)} \quad ....(7)$$

Now, $$\frac{4(1-3Z^2)}{(2Z-1)^2 (Z+1)} = \frac{A}{Z+1} + \frac{B}{2Z-1} + \frac{C}{(2Z-1)^2}$$

$$\Rightarrow \quad 4(1-3Z^2) = A(2Z-1)^2 + B(Z+1)(2Z+1) + C(Z+1) \quad ....(8)$$

put $Z = -1$ in (8) ; $-8 = 9A \Rightarrow A = -\frac{8}{9}$

put $Z = \frac{1}{2}$ in (8) ; $1 = \frac{3}{2}C \Rightarrow C = \frac{2}{3}$

put $Z = 0$ in (8) ; $4 = A - B + C \Rightarrow B = -\frac{8}{9} + \frac{2}{3} - 4$

i.e. $$B = \frac{-8+6-36}{9} = -\frac{38}{9}$$

∴ $$G(a, Z) = \frac{-\frac{8}{9}}{Z+1} - \frac{\frac{38}{9}}{2Z-1} + \frac{\frac{2}{3}}{(2Z-1)^2}$$

Thus, $$a_k = -\frac{8}{9}(-1)^K + \frac{38}{9}2^K + \frac{2}{3}(K+1)2^k.$$

Q 20. State Demorgan' law. Prove it using the truth table. **(PTU, Dec. 2008)**

Solution. Demorgan's Law : Let A and B be any two non empty sets then (i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(i)

A	B	A'	B'	A ∪ B	(A ∪ B)'	A' ∩ B'
1	0	0	1	1	0	0
0	1	1	0	1	0	0
1	1	0	0	1	0	0
0	0	1	1	0	1	1

From last two columns we have $(A \cup B)' = A' \cap B'$

(ii)

A	B	A'	B'	A ∩ B	(A ∩ B)'	A' ∪ B'
1	0	0	1	0	1	1
0	1	1	0	0	1	1
1	1	0	0	1	0	0
0	0	1	1	0	1	1

Since, last two columns of truth table are same

∴ $(A \cap B)' = A' \cup B'$

Q 21. Solve the following recurrence relation using generating function : S (k) – 6S (k – 1) + 5S (k – 2) = 0, k ≥ 2, where S (0) = 1, S (1) = 2. **(PTU, Dec. 2010)**

Solution. The given recurrence relations be ;

$S(k) - 6S(k-1) + 5S(k-2) = 0,\ S(0) = 1,\ S(1) = 2$ ...(1)

Multiply eqn. (1) by Z^k and summing from 2 to ∞.

$$\sum_{K=2}^{\infty} S(K)Z^K - 6\sum_{K=2}^{\infty} S(K-1)Z^K + 5\sum_{K=2}^{\infty} S(K-2)Z^K = 0 \quad ...(2)$$

Now,

$$\sum_{K=2}^{\infty} S(K)Z^K = \left[S(0) + S(1)Z + \sum_{K=2}^{\infty} S(K)Z^K - S(0) - S(1)Z\right]$$

$$= \sum_{K=0}^{\infty} S(K)Z^K - 1 - 2Z$$

$$= G(S:Z) - 1 - 2Z \quad ...(3)$$

$$\sum_{K=2}^{\infty} S(K-1)Z^K = Z\sum_{K=2}^{\infty} S(K-1)Z^{K-1} = Z\sum_{K=1}^{\infty} S(K)Z^K$$

$$= Z\left[S(0) + \sum_{K=1}^{\infty} S(K)Z^K - S(0)\right] = Z[G(S:Z) - 1] \quad ...(4)$$

$$\sum_{K=2}^{\infty} S(K-2)Z^K = Z^2\sum_{K=2}^{\infty} S(K-2)Z^{K-2} = Z^2 G(S:Z) \quad ...(5)$$

putting eqn (3), (4) and (5) in eqn (2) ; we get

$G(S;Z) - 1 - 2Z - 6Z[G(S:Z) - 1] + 5Z^2 G(S:Z) = 0$

$G(S:Z)[5Z^2 - 6Z + 1] = -4Z + 1$

$$G(S:Z) = \frac{1-4Z}{5Z^2 - 6Z + 1} = \frac{1-4Z}{(Z-1)(5Z-1)}$$

$$= \frac{-3/4}{Z-1} - \frac{1/4}{5Z-1}$$

$$S(K) = +\frac{3}{4}(1)^K + \frac{1}{4}(5)^K = \frac{3}{4} + \frac{1}{4}5^K.$$

Q 22. What is the generating function for the sequence : 0, 0, 0, 6, –6, 6, –6, 6, –6, ...? **(PTU, Dec. 2011)**

Solution. Here $a_0 = 0$; $a_1 = 0$; $a_2 = 0$; $a_3 = 6$; $a_4 = -6$; $a_5 = 6$; $a_6 = -6$

$$G(x) = 0 + 0.x + 0.x^2 + 6.x^3 - 6x^4 + 6x^5 - 6x^6 + ...$$

$$= 6x^3(1 - x + x^2 - x^3 .....)$$

$$= \frac{6x^3}{1+x}.$$

Q 23. What is the generating function for the sequences $S_n = ba^n, n \geq 0$? **(PTU, May 2012)**

Solution. Here $S_0 = b, S_1 = ba, S_2 = ba^2, ...$

$$G(x) = b + bax + ba^2x^2 + ...$$

$$= \frac{b}{1-ax}$$ [It forms a G.P with common ratio r = ax, 'a' = b]

Q 24. Solve the recurrence relation $S(n) - 9S(n-1) + 8S(n-2) = 9n + 1$. **(PTU, Dec. 2011)**

Soluton. Given, $S(n) - 9S(n-1) + 8S(n-2) = 9n + 1$ ...(1)

The characteristic eqn is given by putting $S(n) = a^n$ in homogeneous part of eqn (1).

i.e. $a^n - 9a^{n-1} + 8a^{n-2} = 0$

$\Rightarrow a^{n-2}(a^2 - 9a + 8) = 0$

$\Rightarrow \quad a = 1, 8$

$\therefore \quad S_h(n) = a_1 1^n + a_2 8^n$

Now R.H.S of eqn (2) is a polynomial function.

So let the trial solution is of the form $S(n) = n(C_1 + C_2 n)$

putting in eqn. (1) ; we have

$n(C_1 + C_2 n) - 9[(n-1)\{C_1 + C_2(n-1)\}] + 8(n-2)\{C_1 + C_2(n-2)\} = 9n + 1$

Equating the coeffs on both sides, we have

$C_1 - 9C_1 + 18C_2 + 8C_1 - 32C_2 = 9 \Rightarrow -14C_2 = 9$

$$\Rightarrow \quad C_2 = -\frac{9}{14}$$

Also, $9C_1 - 9C_2 - 16C_1 + 32C_2 = 1 \Rightarrow -7C_1 + 23C_2 = 1$

$$\Rightarrow \quad 7C_1 = -\frac{207}{14} - 1 \Rightarrow C_1 = -\frac{221}{98}$$

∴ Complete solution is given

$$S(n) = S_h(n) \quad S_p(n)$$

$$= a_1 + a_2 8^n - \frac{221}{98}n - \frac{9}{14}n^2.$$

Q 25. Solve the recurrence relation $S(K) - 4S(K-1) + 3S(K-2) = K^2$, without using the concept of generating functions. **(PTU, May 2012)**

Solution. Its homogeneous recurrence relation be,

$S(K) - 4S(K-1) + 3S(K-2) = 0$

So its charácteristics equation can be found by $S(K) = a^K$

$\therefore a^K - 4a^{K-1} + 3a^{K-2} = 0$

$\Rightarrow a^{K-2}(a^2 - 4a + 3) = 0 \Rightarrow a = 1, 3$

$\therefore \quad S_h(K) = A_1 1^K + A_2 3^K$

Let the particular solution be $S(K) = C_1 + C_2K + C_3K^2$

putting in given eqn, we have

$[C_1 + C_2K + C_3K^2] - 4[C_1 + C_2(K-1) + C_3(K-1)^2] + 3[C_1 + C_2(K-2) + C_3(K-3)^2] = K^2$

$\Rightarrow 0.C_1 + C_2(K - 4K + 4 + 3K - 6) + C_3[K^2 - 4(K^2 - 2K + 1) + 3(K^2 + 4 - 4K)] = K^2$

$\Rightarrow -2C_2 + C_3(-4K + 8) = K^2$

equating the like coeffs. on both sides, we have

$$-2C_2 + 8C_3 = 0 \; ; -4C_3 = 0 \Rightarrow C_3 = C_2 = 0$$

and C_1 can have any value

∴ Complete solution is given by

$$S(K) = S_h(K) + S_p(K) = A_1 + A_2\, 3^K + C_1.$$

Q 26. Use generating functions to solve the recurrence relation $a_K + 3a_{K-1} - 4a_{K-2} = 0,\ k \ge 2$ with initial conditions $a_0 = 3$ and $a_1 = -2$ and find the sequence which satisfies it. **(PTU, Dec. 20..)**

Solution. Given recurrence relation is,

$$a_K + 3a_{K-1} - 4a_{K-2} = 0,\ k \ge 2 \quad ...(1)$$

Multiply eqn (1) by Z^K and summing from 2 to ∞, we have

$$\sum_{K=2}^{\infty} a_K Z^K + 3\sum_{K=2}^{\infty} a_{K-1} Z^K - 4\sum_{K=2}^{\infty} a_{K-2} Z^K = 0 \quad ...(2)$$

Now,

$$\sum_{K=2}^{\infty} a_K Z^K = \left[a_0 + a_1 Z + \sum_{K=2}^{\infty} a_K Z^K - a_0\, a_1 Z\right]$$

$$= \sum_{K=0}^{\infty} a_K Z^K - 3 + 2Z = G(Z) - 3 + 2Z \quad ...(3)$$

$$\sum_{K=2}^{\infty} a_{K-1} Z^K = Z\sum_{K=2}^{\infty} a_{K-1} Z^{K-1} = Z\sum_{K=1}^{\infty} a_K Z^K = Z\left[a_0 + \sum_{K=1}^{\infty} a_K Z^K - a_0\right]$$

$$= Z[G(Z) - 3] \quad ...(4)$$

$$\sum_{K=2}^{\infty} a_{K-2} Z^K = Z^2\sum_{K=2}^{\infty} a_{K-2} Z^{K-2} = Z^2\sum_{K=0}^{\infty} a_K Z^K = Z^2 G(Z) \quad ...(5)$$

putting (3), (4) and (5) in eqn (2) ; we have

$$[G(Z) - 3 + 2Z] + 3Z[G(Z) - 3] - 4Z^2 G(Z) = 0$$

$$(1 + 3Z - 4Z^2)\, G(Z) = 7Z + 3$$

$$\Rightarrow \quad G(Z) = \frac{7Z+3}{(1+3Z-4Z^2)} = \frac{7Z+3}{(1-Z)(1+4Z)} = \left[\frac{2}{1-Z} + \frac{1}{1+4Z}\right]$$

∴ $a_K = a(1)^K + (-4)^K \Rightarrow a_K = 2 + (-4)^K$ is the required sequence.

Q 27. Find the generating function from the recurrence relation $S(n-2) = S(n-1) + S(n)$ where $S(0) = S(1) = 1,\ n \ge 0$. **(PTU, May 2012)**

Solution. The given recurrence relation can be written as

$$S(n) + S(n-1) - S(n-2) = 0$$

Multiply the given eqn by Z^n and summing from 2 to ∞, we have

$$\sum_{n=2}^{\infty} S(n) Z^n + \sum_{n=2}^{\infty} S(n-1) Z^n - \sum_{n=2}^{\infty} S(n-2) Z^n = 0 \quad ...(1)$$

Now,

$$\sum_{n=2}^{\infty} S(n) Z^n = \left[S(0) + S(1)Z + \sum_{n=2}^{\infty} S(n) Z^n - S(0) - S(1)Z\right]$$

$$= \sum_{n=0}^{\infty} S(n) Z^n - 1 - Z = G(S;Z) - 1 - Z \quad ...(2)$$

$$\sum_{n=2}^{\infty} S(n-1) Z^n = Z\sum_{n=2}^{\infty} S(n-1) Z^{n-1} = \sum_{n=1}^{\infty} S(n) Z^n$$

$$= Z\left[S(0) + \sum_{n=1}^{\infty} S(n) Z^n - 1\right] = Z\left[\sum_{n=0}^{\infty} S(n) Z^n - 1\right] = Z[G(S;Z) - 1] \quad ...(3)$$

$$\sum_{n=2}^{\infty} S(n-2) Z^n = Z^2\sum_{n=2}^{\infty} S(n-2) Z^{n-2} = Z^2\sum_{n=0}^{\infty} S(n) Z^n$$

$$= Z^2 G(S;Z) \quad ...(4)$$

putting (2) ; (3) and (4) in eqn (1) ; we have

$$G(S;Z) - 1 - Z + Z[G(S;Z) - 1] - Z^2 G(S;Z) = 0$$

$$(1 + Z - Z^2)\, G(S;Z) = 1 + 2Z \Rightarrow G(S;Z) = \frac{1+2Z}{1+Z-Z^2}$$

Q 28. Solve the recurrence relation

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3},\ a_0 = 3,\ a_1 = 4,\ a_2 = 12.$$ **(PTU, Dec. 2013)**

Solution. The given recurrence relation is homogeneous substituting $a_n = a^n$ in the given relation, the characteristic equation is given by

$$a^n - 6a^{n-1} + 12a^{n-2} - 8a^{n-3} = 0$$

i.e. $a^{n-3}[a^3 - 6a^2 + 12a - 8] = 0$

$\Rightarrow a^3 - 6a^2 + 12a - 8 = 0 \Rightarrow (a-2)(a^2 - 4a + 4) = 0$

$\Rightarrow (a-2)(a-2)^2 = 0 \Rightarrow a = 2, 2, 2$

∴ $a_n = (C_1 + C_2 n + C_3 n^2)\, 2^n$ ...(1)

Given, $a_0 = 3 \Rightarrow a_0 = 3 = C_1$

$a_1 = 4 \Rightarrow 4 = (C_1 + C_2 + C_3)\, 2 \Rightarrow C_1 + C_2 + C_3 = 2$

i.e. $C_2 + C_3 = -1$ ....(2) [using (1)]

also $a_2 = 12 \Rightarrow 12 = (C_1 + 2C_2 + 4C_3)\, 4 \Rightarrow C_1 + 2C_2 + 4C_3 = 3$

$\Rightarrow C_2 + 2C_3 = 0$ ...(3)

on solving (2) and (3) ; we have

$C_3 = 1$ and $C_2 = -2$

∴ from (1) ; $a_n = (3 - 2n + n^2)\, 2^n$ is te required solution.

□□□

Chapter 7

Group Theory

QUESTION-ANSWERS

Q 1. Show that identity element in a group G is unique. **(PTU, Dec. 2002)**

Solution. Let G be any group equipped with binary operation denoted multiplicatively and let e_1, e_2 are two identifies of G then we shall prove that $e_1 = e_2$

$\forall a \in G, a.e_1 = a$ $(\because e_1$ be the identity of G)

similarly, $a. e_2 = a$ $(\because e_2$ be the identity of G)

$\therefore$ $a. e_1 = a.e_2$

$\Rightarrow$ $e_1 = e_2$ [$\because$ Left concellative laws holds in G]

$\therefore$ Identity in group G is unique.

Q 2. Define normal subgroup of a group G. **(PTU, May 2003 ; Dec. 2007, 2002)**

Solution. A subgroup N of G is said to be normal or invariant or self conjugate subgroup of G if $gN = Ng \ \forall g \in G$ and it is denoted by $N \Delta G$.

e.g. Let $G = S_3 = \{i, (12), (13), (23), (123), (132)\}$

Then, $H = \{i, (132), (123)\}$ is a Normal subgroup of G

as $[G : H] = \dfrac{O(G)}{O(H)} = \dfrac{6}{3} = 2$ $\therefore$ No. of different cosets of H in G in two.

Now, the two right cosets of H in G are H, $H(12) = \{(12), (23), (13)\}$

The two left cosets of H in G are H, $(12)H = \{(12), (23), (13)\}$.

As right cosets of H in G = Left cosets of H in G

$\therefore$ subgroup H is a normal subgroup of G.

Q 3. Define an abelian group. **(PTU, May 2003)**

Solution. A non empty set G equipped with binary operation denoted multiplicative called a group if it satisfies the following axioms.

1. $a. b \in G \ \forall a, b \in G$ (Closure property)
2. $(a.b) c = a. (b.c) \ \forall a, b, c \in G$ (Associatively)
3. $\exists$ an element $e \in G$ s.t. $e. a = a = a.e$ for any $a \in G$
4. $\forall a \in G \ \exists$ an element $b \in G$ s.t $ba = e = ab$

The element e in (3) is called unity or identity of G and the element b in (4) is called inverse a and denoted by a^{-1}.

Further, group (G, .) be commutative or abelian if $a. b = b. a \ \forall a, b \in G$ otherwise it is said to be non-commutative. e.g : (I, +) forms a abelian group.

1. $\forall a, b \in I, a + b \in I$
2. $\forall a, b, c \in I, (a + b) + c = a + (b + c)$
3. $\exists 0 \in I$ s.t. $a + 0 = a + 0 + a$

$\therefore$ 0 behaves as additive identity.

4. $\forall a \in I \ \exists -a \in I$ s.t. $a + (-a) = 0 = (-a) + a$

$\therefore$ $-a$ behaves as additive inverse.

5. $\forall a, b \in I, a + b = b + a$.

$\therefore$ (I, +) forms a abelian group.

Q 4. Define group Homomorphism.

OR

Define isomorphism of groups. **(PTU, May 2007, 2006, 2003)**

Solution. Let G and G' be any two groups. **(PTU, Dec. 2007)**

Then a mapping $\phi : G \to G'$ is called homomorphism iff $\phi(ab) = \phi(a)\phi(b) \ \forall a, b \in G$.

If $\phi : G \to G'$ is onto, Homo then. It is called epimorphism.

If $\phi : G \to G'$ is Homo, I – I then it is called isomorphism or monomorphism.

If ϕ is Homo, onto 1 – I from G to G'. Then the groups are said to be isomorphic and denoted by $G \cong G'$.

If G = G' then the homomorphism is called endomorphism.

If ϕ is also I – I, then the isomorphism is called Automorphism.

e.g : The mapping $\phi : G \to G'$ defined by $\phi(a) = e' \ \forall a \in G$

$\forall a, b \in G, \phi(ab) = e' = e' . e' = \phi(a)\phi(b)$

where e' be the identity element of G'.

$\therefore$ ϕ is homomorphism called zero-homomorphism.

Q 5. Define semi-group with examples. **(PTU, Dec. 2009 ; May 2009, 2004)**

Solution. A non empty set G equipped with binary operation denoted multicatively called a semi-group if it satisfies following two properties.

(i) $a. b \in G \ \forall a, b \in G$ (Closure property)

(ii) $(a.b) c = a. (b.c) \ \forall a, b, c \in G$ (Associativity)

e.g : $N = \{1, 2, 3, ...\}$ forms a semi group under addition.

Q 6. Define group with examples.

OR **(PTU, May 2004)**

What is a group? **(PTU, May 2009 ; Dec. 2005)**

Solution. A non empty set G equipped with binary operation denoted multiplicative called a group if it satisfies the following axioms.

1. $a. b \in G \ \forall a, b \in G$ (Closure property)
2. $(a.b) c = a. (b.c) \ \forall a, b, c \in G$ (Associatively)

3. $\exists$ an element $e \in G$ s.t. $e.a = a = a.e$ for any $a \in G$

4. $\forall a \in G \; \exists$ an element $b \in G$ s.t $ba = e = ab$

The element e in (3) is called unity or identity of G and the element b in (4) is called inverse a and denoted by a^{-1}.

e.g. (I, +) form an abelian group, where I = set of integers.

Solution. 1. $\forall a, b \in I, a + b \in I$

2. $\forall a, b, c \in I, (a + b) + c = a + (b + c)$

3. $\exists\, 0 \in I$ s.t. $a + 0 = a = 0 + a$

$\therefore$ 0 behaves as additive identity.

4. $\forall a \in I \; \exists -a \in I$ s.t. $a + (-a) = 0 = (-a) + a$

$\therefore$ –a behaves as additive inverse.

5. $\forall a, b \in I, a + b = b + a$.

Q 7. Define subgroup. **(PTU, May 2007, 2006)**

Solution. A non-empty subset H of G (group) is called subgroup of G if H is a group in itself under the borrowed operation of G. {e} and G are improper or trivial subgroups of G while all other subgroups of G are called proper subgroups of G.

e.g. Now $Q \subset R$, (Q, +) is a subgroup of (R, +) while (Q^+, +) is not a subgroup of (R, +). Here $Q^+ \subseteq R$.

Because inverses of elements of Q^+ does not exist in R.

e.g. Let n be any fixed integer. Now $nZ = \{0, n - n, 2n, -2n ....\}$ is a proper subgroup of (Z, +) when $n \neq 0, 1 -1$.

e.g $J_6 = \{0, 1, 2, 3, 4, 5\}$ is a group under addition modulo 6.

Now $H_1 = \{0, 3\} \subset J_6$ further H_1 is a group in itself as {0} behaves as identity element and 3 has inverse {3}.

$\therefore H_1$ is a proper subgroup of J_6.

Further $H_2 = \{0, 2, 4\} \subset J_6$. Now 2, 4 has inverse 4, 2.

Futher Associative laws holds and {0} be the identity element. $\therefore H_2$ be the proper subgroup of J_6.

Q 8. If a, b, c are elements of a group G and a∗b = c∗a, then b = c? Explain your answer. **(PTU, Dec. 2006)**

OR

True or False. If a, b, c are elements of a group G and a* b = c*a, then b = c? Explain your answer. **(PTU, May 2005)**

Solution. If $a, b, c \in G$ and $a*b = c*a$ ...(1)

Now $c*a = a*c$ if G is abelian

$\therefore$ $a*b = a*c$ [unless G is abelian]

$\Rightarrow$ $b = c$ [Right concellation law holds]

Q 9. Give an example of a finite group. **(PTU, May 2010)**

Solution. Define an operation called addition of residue classes modulo 5 or $I/_{(5)}$

i.e. $[i] + [j] = [i + j] \; \forall [i], [j] \in I/_{(5)}$.

Now $I/_{(5)}$'s the set of all residue classes of remainder by dividing an arbitrary residue classes of integers by 5.

The composition table is given below :

+modulo 5	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

Clearly $I/_{(5)}$ is closed, associative laws also holds. Also from composition table it is symmetrical about main diagonal. $\therefore$ It is commutative, also [0] behave as identity element. Further, the elements [1], [2], [3]. [4] have inverses are [4], [3], [2], [1] resp.

$\therefore I/_{(5)}$ forms a group (abelian) under additive modulo 5.

Q 10. Let G be any group and let a be any element of G. Define the cyclic group generated by a. **(PTU, Dec. 2002)**

Solution. Let a be any arbirary element of G. Then the set $H = \{a^n \; \forall n \in I\}$ is a subgroups of G as $a \in G \Rightarrow a^n \in G \Rightarrow H \subset G$ further $a = a^1 \in H$

$\therefore$ H is non-empty subset of G.

Let $x, y \in H$ where $x = a^m, y = a^n$ where $m, n \in I$

$$xy = a^m . a^n = a^{m+n} \in H \text{ as } m + n \in I$$

and $$x^{-1} = (a^m)^{-1} = a^{-m} \in H \text{ as } -m \in I$$

$\therefore$ H is a subgroup of G. So H is the smallest subgroup of G generated by a. So H be the cyclic subgroup of G generated by a and denoted by H = <a>.

Cyclic Group : A group G is said to be cyclic or monic generated by a if every element of G can be expressed in powers of 'a' and denoted by G = <a>

i.e. $\forall x \in G$ and $G = \langle a \rangle \Rightarrow x = a^n$ where $n \in I$

where a is called generator of G.

If $G = \langle a \rangle \Rightarrow \forall x \in G = \langle a \rangle \Rightarrow x = a^n = (a^{-1})^{-n}$ where $n \in I$

$\therefore G = \langle a^{-1} \rangle$ So a cyclic group may have more than one generator.

e.g 1. (I, +) is a cyclic group.

Since every integer can be of the form K.1 where $K \in I$ So 1 be the generator of (I, +). Similarly –1 can also be its generator. Hence (I, +) is cyclic.

e.g 2. G = {1, w, w²} forms a cyclic group.

Since O (G) = 3, w be an imaginary cube root of unity. Since $w^3 = 1, w^1 = w, w^2 = w^2$

So G = <w>

Similarly $w^2 = w^2$; $(w^2)^2 = w^4 = w^3 . w = w, (w^2)^3 = (w^3)^2 = w^3$

So G = $\langle w^2 \rangle$

Hence G is cyclic.

Q 11. Let H be a subgroup of G. Define a coset representative system for H in G. (PTU, May 2008)

OR

Define cosets with examples. (PTU, May 2005, 2004)

OR

What do you mean by cosets? (PTU, Dec. 2006)

Solution. Since, H be a subgroup of group G and $a \in G$.

Then the set $Ha = \{ha : \forall h \in H\}$ is called right coset of H in G determined by a.

and the set $aH = \{ah : \forall h \in H\}$ is called left coset of H in G determined by a.

Now G is abelian, $Ha = aH$

Now $e \in H$, H is subgroup, $a = ea \in Ha$

Further, if G is a additive group.

Then right coset of H in G determined by a

$$H + a = \{h + a;\ \forall h \in H\}$$

and left cosets of H in G determined by a

$$a + H = \{a + h;\ \forall h \in H\}.$$

e.g. : Z be a group of integers under addition and 5Z is a set of multiple of '5' is a subgroup of Z.

Now
$$5Z + 1 = \{\ldots\ldots, -9, -4, 1, 6, 11, 16, \ldots\ldots\} = \{5n + 1 ; n \in Z\}$$
$$5Z + 2 = \{\ldots\ldots, -8, -3, 2, 7, 12, \ldots\ldots\} = \{5n + 2 ; n \in Z\}$$
$$5Z + 3 = \{\ldots\ldots, -7, -2, 3, 8, 13, \ldots\ldots\} = \{5n + 3 ; n \in Z\}$$
$$5Z + 4 = \{\ldots\ldots, -6, -1, 4, 9, 14, \ldots\ldots\} = \{5n + 4 ; n \in Z\}$$
$$5Z + 5 = \{\ldots\ldots, -5, 0, 5, 10, \ldots\ldots\} = \{5n + 5 ; n \in Z\}$$

These are the distinct right cosets of 5Z in Z.

Similarly different left cosets of 5Z in Z are

$$1 + 5Z = \{1 + 5n ; n \in N\}$$
$$2 + 5Z = \{2 + 5n ; n \in N\}$$
$$3 + 5Z = \{3 + 5n ; n \in N\}$$
$$4 + 5Z = \{4 + 5n ; n \in N\}$$
$$5 + 5Z = \{5n ; n \in N\}.$$

Q 12. Let S be a semigroup with an identity element e and suppose b and b are inverses of an element a in S. Show that b = b. (PTU, Dec. 2003)

Solution. Let S be a subgroup equipped with binary operation denoted by '*' so closure and associativity holds under *, further e be the identity of S.

Further for some $a \in S$, given b, b' be an inverses in S

$$a * b = e = b * a \qquad \ldots(1)$$

and
$$a * b' = e = b' * a \qquad \ldots(2)$$

Let
$$b * (a * b') = (b * a) * b' \qquad \text{[Associative law holds]}$$

$$\Rightarrow \quad b * e = e * b' \qquad [\because S \text{ is with identity } e]$$
$$\Rightarrow \quad b = b'$$

Hence the result.

Q 13. Prove that every subgroup of a cyclic group is cyclic. (PTU, Dec. 2004)

OR

What do you mean by cyclic group? Show that any subgroup of a cyclic group is cyclic. (PTU, Dec. 2009)

Solution. Let G be any cyclic group so by def. $\exists$ an element $a \in G$ s.t. $G = \langle a \rangle$ and let H be the subgroup of G. Now we want to prove that H is cyclic.

If $H = \{e\}$ then $H = \langle e \rangle \therefore$ H is cyclic.

So let us suppose that $H \neq \{e\} \therefore O(H) > 1$

Let $x \neq e \in H \subset G \Rightarrow x \in G = \langle a \rangle \Rightarrow x = a^n$ for some $n \in I$

further as $x \in H$, $x^{-1} \in H \Rightarrow a^{-n} \in H$

So H contains some element with positive integral powers of a.

Let m be the least positive integer s.t. $a^m \in H$.

Now we shall prove that $H = \langle a^m \rangle$

Let $y \in H \subset G \Rightarrow y = a^k$ for some $k \in I$ and let $k = mq + r$ where $0 \le r < m$

Also $a^r = a^{k-mq} = a^k (a^m)^{-q} \in H$ $(\because a^k \in H, a^m \in H)$

but $0 \le r < m$ s.t. $a^r \in H \therefore r = 0$

Hence $k = mq$ therefore $y = a^k = a^{mq} = (a^m)^q$ for some $q \in I$.

$\therefore H = \langle a^m \rangle \Rightarrow$ H is cyclic.

Hence every subgroup of cyclic group is cyclic.

Q 14. Prove that for any commutative monoid (M,*), the set of idempotent elements of M form a submonoid. (PTU, Dec. 2004)

Proof : Let (S, +) be any semigroup. Then $a \in S$ is said to be idempotent if $a^2 = a$.

Let (M, *) be a commutative monoid with identity e and let T be the set of all idempotent elements of M and we want to prove that T forms a submonoid. Now $e * e = e \Rightarrow e \in T$. Now we want to prove that $\forall\ a, b \in T$, $a * b \in T$.

For this,
$$(a * b) * (a * b) = a * (b * a) * b \qquad (\because M \text{ is associative})$$
$$= a * (a * b) * b \qquad (\because M \text{ is commutative})$$
$$= (a * a) * (b * b) \qquad (\because M \text{ is associative})$$
$$= a * b \qquad (\because a, b \in T \therefore a, b \text{ are idempotent})$$

$\therefore a * b \in T$.

Q 15. Let (G, *) be a group and a be an element of G. Define $f : G \to G$ by $f(x) = a * x$.

(a) Prove that f is bijection.

(b) On the basis of part (a), describe a set of bijection on set of integers. (PTU, May 2005)

Solution. (a) Let (G, *) be a group and a be any element of G and $f : G \to G$ defined by $f(x) = a * x$

$\forall$ x, y $\in$ G s.t f (x) = f (y)

$\Rightarrow$ a*x = a*y

$\Rightarrow$ x = y [Concellation laws holds in group]

$\therefore$ f is one-one

for any x $\in$G, a $\in$ G $\Rightarrow$ a* x $\in$G as closure property holds in a group s.t f(x) = a*x

$\therefore$ f is onto

Hence f is 1 – 1 and onto $\therefore$ f is bijection.

(b) Let I be the set of integers and define

f : I $\to$ I by f (x) = a + x $\forall$ x $\in$I, a $\in$I

$\forall$ x, y $\in$ I s.t f (x) = f (y) $\Rightarrow$ a + x = a + y

$\Rightarrow$ x = y $\Rightarrow$ f is 1 – 1

for any y $\in$ I $\exists$ y – a $\in$I as y, a $\in$I $\Rightarrow$ y – a $\in$ I

s.t f (y – a) = a + y – a = y $\Rightarrow$ f is onto.

$\Rightarrow$ f is 1 – 1, onto, thus f is bijection.

Q 16. How group theory is applied in coding theory? **(PTU, Dec. 2005)**

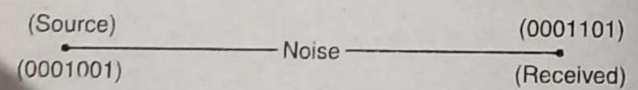
Solution. Application of Group Theory in Coding Theory : By using group theory we can find the solutions of so many coding problems. A coding problem is a problem which is used to represent distinct messages by means of a sequence of letters from a given alphabet. A sequence of letter from an alphabet is called a word. A code is a collection of word that is used to represent distinct messages. A word in a code is called **codeword**.

(i) In error correction : We know that, in a simple communication model, there are three essential parts :

1. Source (Transmitter)
2. Communication Channel
3. Destination (Receiver)

A message or information originates from source, passes over a communication channel and reaches destination. Since, a communication channel is subjected to variety of disturbances the message may get distorted.

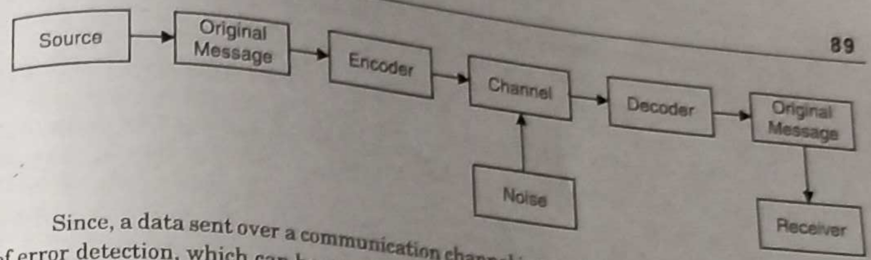
Noise : Any such disturbance which can distort the sent message is called noise.



The main aim of any communication system is to minimize the distortion due to noise and to recover the original message in same manner. The efficiency of a communication channel can be improved by a device called encoder.

Encoder : It is a device which transforms the message in such a way that the presence of noise on the transformed message can be detected.

Decoder : It is a device which transforms the encoded message into its original form.



Since, a data sent over a communication channel is prove to noise hence there are methods of error detection, which can be used for the correction of these errors. Any error which needs correction has to be detected first. Now we define one more important function.

Encoding Function : A one to one function e : $B^m \to B^n$ for which we can choose +ve integer n > m and denoted by (m, n) encoding function. If b $\in$ B^n then e (b) is code word. The number of 1's in code word x is called weight of x and is denoted by | x |.

By using the minimum-distance decoding criteria, it has been proved that a code of distance 2k + 1 can correct k or lesser transmission error.

(ii) Group codes : Let e : $B^m \to B^n$ be any encoding function that it is said to be group code if [e(b^n) s.t. b$\in B^m$] = e(B^m) is a subgroup of group (B^n, $\oplus$) and it is denoted by N.

By using minimum distance decoding criterion we can determine the transmitted word corresponding to a received word. Let (B^n i.e. (N, $\oplus$)) is a group code. Let y be a received word and d (x_i,y) = The distance between and x_i and y = $x_i \oplus$ y.

Also the weight of the word is the coset $B^n \oplus$ y are the distances between the code words in B^n and y. Let e denotes one of the words of smallest weight. Then, according to the minimum distance decoding criterion, e $\oplus$ y = x_i is the transmitted codeword. Thus, by using the axioms of group theory, we can find the transmitted codewords.

Q 17. Prove that every cyclic group is abelian. **(PTU, May 2006)**

OR

If {G, *} is cyclic, then it is abelian. **(PTU, May 2005)**

Solution. Let G be the cylic group generated by a i.e. G = <a>

$\forall$ x, y $\in$ G = <a> $\Rightarrow$ x = a^m, y = a^n for some m, n $\in$ I

$\therefore$ xy = a^m. a^n = a^{m+n} = a^{n+m} = a^n. a^m = yx

$\Rightarrow$ G is abelian.

Q 18. Prove that intersection of two normal subgroups is again a normal subgroup. **(PTU, May 2007, 2006)**

OR

If H and K are two subgroups of a group (G, .), prove that H $\cap$ K is a subgroup of G. **(PTU, May 2010 ; Dec. 2007)**

Solution. Let H, K are normal subgroups of a group G.

Now we want to prove that H $\cap$ K is a normal subgroup of G, first of all, we prove that H $\cap$ K is a subgroup of G.

As H $\subseteq$ G, K $\subseteq$ G $\Rightarrow$ H $\cap$ K $\subseteq$ G

Further e $\in$H, e $\in$K $\Rightarrow$ e $\in$H $\cap$ K

$\therefore H \cap K$ is a non-empty subset of G.

$\forall$ a, b $\in H \cap K \Rightarrow$ a, b $\in$ H and a, b $\in$ K

$\Rightarrow ab, a^{-1} \in H$ and $ab, a^{-1} \in K \Rightarrow ab, a^{-1} \in H \cap K$

$\Rightarrow H \cap K$ is a subgroup of G.

$\forall g \in G$ and $x \in H \cap K \Rightarrow x \in H$ and $x \in K$

Now $\forall g \in G$, $x \in H$ and H is a normal subgroup of G.

$\therefore \quad gxg^{-1} \in H$

$\forall g \in G$, $x \in K$ and K is a normal subgroup of G $\therefore gxg^{-1} \in K$ ....(1)

$\therefore gxg^{-1} \in H \cap K$ [using (1) and (2)] ....(2)

Thus $H \cap K$ is a normal subgroup of G.

Q 19. If H is a subgroup of G. Show that HH = H. **(PTU, Dec. 2002)**

Solution. A non-empty subset H of group G is a subgroup of G iff HH = H.

Let H be a subgroup of G T.P. HH = H

Now $\forall h \in H$, $h = he \in HH$ (as $e \in H$) $\therefore H \subseteq HH$

further $h_1, h_2 \in HH \ \forall h_1, h_2 \in H$ Now H is a subgroup of G. ....(1)

$\therefore h_1 h_2 \in H \Rightarrow HH \subseteq H$

$\therefore$ from (1) and (2) we have HH = H.

Q 20. What is a congruence relation on a subgroup? Explain. **(PTU, May 2005)**

Solution. Let H be any subgroup of group G and define a congruence relation on H

$\forall a, b \in H$, $a R b \Leftrightarrow a \equiv b \pmod{H} \Leftrightarrow ab^{-1} \in H$

Now we want to prove that congruence relation is an equivalence relation on H.

(i) Reflexive : $\forall a \in H$ as H is a subgroups of G $\therefore a^{-1} \in H$

$\therefore a, a^{-1} \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$

[$\because$ H is a subgroup of G $\therefore$ H is a group in itself]

Now $aa^{-1} \in H \Rightarrow a = a \pmod{H} \Rightarrow a Ra$

$\therefore$ R is reflexive.

(ii) Symmetric : Let a, b $\in$ H, a R b $\Rightarrow a \equiv b \pmod{H} \Rightarrow ab^{-1} \in H$

as $a \in H$, H be a subgroup of G $\Rightarrow a^{-1} \in H$

thus $(ab^{-1})^{-1} \in H \Rightarrow (b^{-1})^{-1} a^{-1} \in H \Rightarrow ba^{-1} \in H$

$\Rightarrow \quad b \equiv a \pmod{H} \Rightarrow bRa$ [$\because (b^{-1})^{-1} = b$]

$\therefore$ R is symmetric.

(iii) Transistive : Let a, b, c $\in$ H, s.t a R b, bRc than we want to prove that aRc

Now $\quad a R b \Rightarrow a \equiv b \pmod{H} \Rightarrow ab^{-1} \in H$ ....(1)

$\quad b Rc \Rightarrow b \equiv c \pmod{H} \Rightarrow bc^{-1} \in H$ ....(2)

Now $ab^{-1}, bc^{-1} \in H \Rightarrow (ab^{-1})(bc^{-1}) \in H$

[$\because$ If a, b $\in$ H $\Rightarrow$ ab $\in$ H as H is a subgroup of G]

$\Rightarrow a(b^{-1}b)c^{-1} \in H \Rightarrow aec^{-1} \in H$ [$\because b^{-1}b = e$]

$\Rightarrow ac^{-1} \in H \Rightarrow a \equiv c \pmod{H} \Rightarrow aRc$

$\therefore$ R is transitive.

Thus R is reflexive, symmetric and transistive.

$\therefore$ R is a equivalence relation on H.

Q 21. State and prove Lagrange's theorem on finite groups.
(PTU, Dec. 2007, 2006, 2003 ; May 2012, 2006, 2004)

Solution. Statement of Lagrange's Theorem :
If H be a subgroup of a finite groupe G, then O (H)/O (G)

Proof : For proving this theorem, first of all we prove two lemmas.

Lemma 1 : If H be a subgroup of group G then the relation congurence modulo H is an equivalence relation on G and [a] = Ha for any $a \in G$.

Proof : Let e be an identity element of H as $aa^{-1} = e \in H$

$\therefore a \equiv a \pmod{H}$ $\therefore$ relation is reflexive.

Further as $a = b \pmod{H} \Rightarrow ab^{-1} \in H$

Now H is a subgroup of G if $a \in H$ then $a^{-1} \in H$

$\Rightarrow (ab^{-1})^{-1} \in H \Rightarrow (b^{-1})^{-1} a^{-1} \in H \Rightarrow ba^{-1} \in H$

$\Rightarrow b \equiv a \pmod{H}$ $\therefore$ relation is symmetric.

Further if $a \equiv b \pmod{H}$ and $b = c \pmod{H}$

then we have $ab^{-1} \in H$ and $bc^{-1} \in H$

Now $(ab^{-1})(bc^{-1}) \in H$ [$\because$ If H is a subgroup of G then ab $\in$ H for any a, b $\in$ H]

$\Rightarrow ac^{-1} \in H$ $\therefore$ relation is transitive.

$\therefore$ Relation congruence modulo H is an equivalence relation on G. So G can be decomposite to mutually disjoint equivalence classes.

Further Let $\forall x \in [a] \Rightarrow x \equiv a \pmod{H} \Rightarrow xa^{-1} \in H \Rightarrow x \in Ha$

$\Rightarrow \quad [a] \subseteq Ha$

$\forall y \in Ha \Rightarrow y = ha$ where $h \in H$ ....(1)

$\Rightarrow \quad ya^{-1} = h \in H \Rightarrow y \equiv a \pmod{H} \Rightarrow y \in [a]$

$\therefore \quad Ha \subseteq [a]$

From (1) and (2) ; Ha = [a] ....(2)

Lemma 2 : Prove that there is a 1–1 correspondence between any two right cosets of H in G.

Proof : Let Ha, Hb are the two right cosets of H in G.

Define a map $\psi : Ha \to Hb$

by $\psi(ha) = hb \ \forall h \in H$.

Now we want to prove that ψ is one-one and onto.

For onto, for any $hb \in Hb \ \exists \ ha \in Ha$ s.t. $\psi(ha) = hb$

For 1 – 1, Let $\psi(h_1 a) = \psi(h_2 a)$ (where $h_1, h_2 \in H$)

$\Rightarrow \quad h_1 b = h_2 b$ (cancellation law)

$\Rightarrow \quad h_1 = h_2$

$\Rightarrow \quad h_1 a = h_2 a$

$\therefore \psi$ is 1 – 1, onto.

Further He = H $\therefore$ H itself is a right coset of H in G.

Thus there is an 1–1 correspondence between any cosets Ha and H. If H is finite then O (Ha) = O (H).

Let O (G) = n and O (H) = m, we want to prove that m/n. Using lemma 1 and 2 it follows that, G decomposes into mutually disjoint right cosets of H in G, each of which has order O (H) *i.e.* m elements, if G decomposes into k different right cosets of G then n = mk $\therefore$ m/n.

Q 22. Prove that the Kernel of a Homomorphism f from (G, .) to group (G', .) is a normal subgroup of (G, .). **(PTU, Dec. 2007)**

Solution. Given, ϕ is a homomorphism from G into G' with Kernal K. Then we want to prove that K is a normal subgroup of G.

Since $\phi(e) = e' \Rightarrow e \in K \therefore$ K is a non-empty subset of G.

$\forall x, y \in K \Rightarrow \phi(x) = e' = \phi(y)$

Let $\phi(xy) = \phi(x)\,\phi(y) = e' . e' = e' \Rightarrow xy \in K$.

$\forall x \in K$, Let $\phi(x^{-1}) = [\phi(x)]^{-1} = (e')^{-1} = e' \Rightarrow x^{-1} \in K$

$\therefore$ K be a subgroup of G.

Let $g \in G$ and $x \in K$ we want to prove that $g x g^{-1} \in K$.

For this let $\phi(g x g^{-1}) = \phi(g)\,\phi(x)\,\phi(g^{-1})$ as ϕ is Homomorphism

i.e. $\phi(g x g^{-1}) = \phi(g)\, e'\, \phi(g^{-1}) = \phi(g)\,\phi(g^{-1}) = \phi(g g^{-1}) = \phi(e) = e'$

$\Rightarrow g x g^{-1} \in K \therefore$ K be a normal subgroup of G.

Q 23. Prove that the set G = {1, 2, 3, 4, 5, 6} is a finite alielian groups of order 6 with respect to multiplication modulo 7. **(PTU, May 2009)**

OR

Find the multiplication table for G = {1, 2, 3, 4, 5, 6} under multiplication modulo 7. **(PTU, Dec. 2009)**

Solution. Clearly G is closed under multiplication module 7 and also table is symmetrical about main diagonal $\therefore$ G is abelian and also associative laws holds.

module 7

.	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Futher, 1 behave as identity element of G.

Clearly from table

The inverses of 1, 2, 3, 4, 5, 6 composition table are 1, 4, 5, 2, 3, 6 respectively.

$\therefore$ G forms a abelian group of order 6 under multiplication modulo 7.

Q 24. Consider the group G = {1, 2, 3, 4, 5, 6} under multiplication modulo 7.

(a) Find the multiplication table of G.

(b) Prove that G is a group.

(c) Find 2^{-1}, 3^{-1} and 1^{-1}.

(d) Find the orders and subgroups generated by 2 and 3.

(e) Is G cyclic. Justify your answer. **(PTU, Dec. 2009)**

Solution. (a) The composition table for group G = {1, 2, 3, 4, 5, 6} is given as under

. modulo 7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(b) Since $2.4 \equiv 1 \pmod 7$ and $3.5 \equiv 1 \pmod 7$ also $6.6 \equiv 1 \pmod 7 \therefore 4, 5, 6$ are inverses of 2, 3, 6 *i.e.* $2^{-1} = 4$; $3^{-1} = 5$ and $6^{-1} = 6$

(c) Since, $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 1 \pmod 7 \therefore$ ord (2) = 3 group generated by 2 is $\{2, 2^2, 2^3\}$ *i.e.* {1, 2, 4} and group generated by 3 is {1, 2, 3, 4, 5, 6} = G $[\because 3^1 = 3 \pmod 7 ; 3^2 \equiv 2 \pmod 7 ; 3^3 \equiv 6 \pmod 7$ $3^4 \equiv 3 \pmod 7 ; 3^5 = 4 \pmod 7 ; 3^6 \equiv 1 \pmod 7]$

(d) Since O (3) = 6 = O (G) $\therefore$ G is cyclic group generated by 3 *i.e.* G = <3>.

Q 25. Define semi group and monoid.

OR **(PTU, May 2011)**

Define monoid, semigroup, group and rings with examples. **(PTU, May 2008)**

Solution. Monoid : A non empty set 'A' equipped with binary operation * is said to form Monoid. If it satisfies following properties.

1. $(a*b)*c = a*(b*c)\ \forall\ a, b, c \in A$
2. for any $a \in A\ \exists\ e \in A$ s.t. $e*a = a = a*e$
   e is called identity element of A.

e.g. The set N of natural numbers is a semi-group under the operation *defined by x*y = max {x, y} is forms a monoid?

$\forall x, y, z \in N, (x * y) * z = \max\{\max\{x, y\}, z\} = \max\{x, y, z\}$

Similarly, $x * (y * z) = \max\{x, \max(y, z)\}$
$= \max(x, y, z)$

$\therefore \quad (x * y) * z = x * (y * z)$

$\Rightarrow$ * is associative. Thus (N, *) is semigroup.

Further for any x we have

$x * 0 = \max\{x, 0\} = x$

and $0 * x = \max\{0, x\} = x$

where 0 be the identity element.

$\therefore$ (N, *) forms monoid.

Semigroup : A non empty set G equipped with binary operation denoted multicatively called a semi-group if it satisfies following two properties.

(i) $a.b \in G\ \forall\ a, b \in G$ (Closure property)

(ii) $(a.b)\,c = a.(b.c)\ \forall\ a, b, c \in G$ (Associativity)

e.g : N = {1, 2, 3, ...} forms a semi group under addition.

Group : A non empty set G equipped with binary operation denoted multiplicative called a group if it satisfies the following axioms.

1. $a . b \in G \; \forall a, b \in G$ (Closure property)
2. $(a.b)\, c = a.\, (b.c) \; \forall \; a, b, c \in G$ (Associatively)
3. $\exists$ an element $e \in G$ s.t. $e.\, a = a = a.e$ for any $a \in G$
4. $\forall a \in G \; \exists$ an element $b \in G$ s.t $ba = e = ab$

The element e in (3) is called unity or identity of G and the element b in (4) is called inverse a and denoted by a^{-1}.

e.g. (I, +) form an abelian group, where I = set of integers.

1. $\forall a, b \in I, a + b \in I$
2. $\forall a, b, c \in I, (a + b) + c = a + (b + c)$
3. $\exists\, 0 \in I$ s.t. $a + 0 = a = 0 + a$
   $\therefore$ 0 behaves as additive identity.
4. $\forall a \in I \; \exists -a \in I$ s.t. $a + (-a) = 0 = (-a) + a$
   $\therefore$ –a behaves as additive inverse.
5. $\forall a, b \in I, a + b = b + a.$

Ring : It is the algebraic system with two binary operations denoted '+' and '.' resp.

A non empty set R equipped with two binary operations denoted additively '+' and multiplicatively '.' is called a ring and satisfies following axions. $\forall$ a, bn, $c \in R$

1. $a + b \in R$
2. $(a + b) + c = a + (b + c)$
3. For any $a \in R \; \exists\, 0 \in R$ s.t. $a + 0 = a = 0 + a$
4. For any $a \in R \; \exists\, b \in R$ s.t. $b + a = 0 = a + b$
5. $a + b = b + a$
6. $a . b \in R$
7. $(a. b). c = a. (b. c)$
8. $a. (b + c) = a . b + a . c$ (Distributive law)

A ring is said to be commutative or abelian ring if $\forall a, b \in R$, ab = ba.

e.g. I/<4> = {[0], [1], [2], [3]}

The composite table is given as under :

+4	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[2]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

.4	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

Clearly it is a ring under + modulo 4 and ·modulo 4. Also it symmetrical about main diagonal. $\therefore$ It is commutative. So it is a finite commutative ring with unity [1]. Now here we have $[2] \neq [0] \in$ I/<4> but [2]. [2] = [0] (mod 4).

$\therefore$ I/<4> has proper zero divisor. $\therefore$ It is not an integral domain.

Q 26. Show that a semi group with two idempotent elements can not be a group. **(PTU, Dec. 2010)**

Solution. Let if possible, the given semi group G is a group and let a, b are two distinct elements of G,

s.t. $a^2 = a$ and $b^2 = b$

further e be any identity of G s.t $ae = a$

$\therefore a^2 = a = ae \Rightarrow a.a = a.e \Rightarrow a = e$ (using left concellative law)

Similarly b = e but identity of G always be unique $\therefore$ a = b but a and b are distinct.

$\therefore$ Our supposition is wrong. Thus G cannot forms a group.

Q 27. Let G be a finite group with identity element e. Show that $a^n = e$ for any $a \in G$. **(PTU, Dec. 2010)**

Ans. Let G be a finite group, $\forall a \in G$.

Consider the set $S = \{a, a^2, a^3, ...\} \Rightarrow S \subset G$

as G is finite $\therefore$ S is also finite.

Let $a^i = a^j$ where $i \neq j$ Let $i > j \Rightarrow i - j > 0$

$\Rightarrow a^{i-j} = e \therefore \exists$ a + ve integer i – j when raised as power of a gives identity element $\therefore$ by well ordering principle set of the integers must have smallest element. Let n be the smallest positive integer s.t $a^n = e$.

Q 28. Let (G, o) be a group. Show that (G, o) is an abelian group if and only if $(a \text{ o } b)^2 = a^2 \text{ o } b^2$. **(PTU, Dec. 2010)**

Ans. $(ab)^2 = a^2b^2$ T.P.G. is abelian

$\Rightarrow$ ab ab = aa bb

$\Rightarrow$ bab = abb

$\Rightarrow$ ba = ab $\Rightarrow$ G is abelian

Conversly : G is abelian i.e. ab = ba T.P. $(ab)^2 = a^2 b^2$

Now $(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = (aa)(bb) = a^2 b^2$

Q 29. Let G be direction group of two by two invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; ad – bc ≠ 0. Let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \neq 0 \right\}$. Show that H is a normal subgroup of G. **(PTU, Dec. 2010)**

Solution. Clearly $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H \therefore$ H is a non empty subset of G.

$\forall \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \in H$, where $a \neq 0, b \neq 0$

s.t. $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \in H$ [as $a \neq 0, b \neq 0$ i.e. $ab \neq 0$]

Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in H$ s.t $A^{-1} = \frac{1}{a^2}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2} & 0 \\ 0 & \frac{a}{a^2} \end{bmatrix} \in H$

$\left[\text{as } a \neq 0, \frac{a}{a^2} \neq 0\right]$

$\therefore$ H be a subgroup of G.

Let $h = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in H, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ where $a \neq 0$, $ab \neq 0$ and $b \neq 0$

$$\therefore \quad gh\,g^{-1} = \frac{1}{ad - bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in H$$

$\therefore$ H is normal in G.

Q 30. Suppose $f : G \to G'$ is a group homomorphism. Prove that $f(e) = e'$ and $f(a^{-1}) = f(a)^{-1}$. **(PTU, May 2011)**

Solution. If f is a homomorphism from a group G into a group G'. Then $f(e) = e'$ where e, e' are identity element of G and G' and $f(a^{-1}) = [f(a)]^{-1} \; \forall \, a \in G$.

Proof : For any $a \in G$, $a = ae \Rightarrow f(a) = f(ae)$ and f is Homo.

$$f(a) = f(a)\,f(e) \text{ So } f(e) \text{ be the identity element of G}$$

$$\therefore \quad f(e) = e'$$

For second part we have $aa^{-1} = e \; \forall a \in G$

$$f(aa^{-1}) = f(e) = e' \Rightarrow f(a)\,f(a^{-1}) = e'$$

$\therefore$ by def. of inverse in G' we have $f(a^{-1}) = [f(a)]^{-1}$.

Kernal : If f be homomorphism from a group G into group G'. Then Kernel of f is denoted *k* and given by

$$k = \{x, x \in G : f(x) = e'\}$$

Q 31. In any group G, prove that, $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$. **(PTU, Dec. 2011)**

Solution. Since $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = ae\,a^{-1}$ $(\because bb^{-1} = e)$

$= aa^{-1} = e$ $(\because$ G is a group)

also $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)\,b$

$= b^{-1}eb$ $(\because a^{-1}a = e)$

$= b^{-1}b = e$ $(\because b^{-1}b = e)$

$\Rightarrow \quad (ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

$\Rightarrow \quad (ab)^{-1} = b^{-1}a^{-1}$. [by def. of inverse]

Q 32. If $a^{-1} = a \; \forall \, a \in G$, where G is a group, then show that G is commutative. **(PTU, May 2012)**

Solution. $\forall \, a \in G$, $b \in G \Rightarrow a^{-1} = a$, $b^{-1} = b$ ....(1)

$\therefore ab \in G \Rightarrow (ab)^{-1} = ab \Rightarrow b^{-1}a^{-1} = ab$ (reversal law)

$\Rightarrow ba = ab \Rightarrow$ G is abelian.

Q 33. Show that a non-empty subset H of a group G is a subgroup of the group G if and only if, $ab^{-1} \in H \; \forall \, a, b \in H$. **(PTU, Dec. 2011)**

Solution. Let H be a subgroup of G. $\therefore$ H is a group of itself.

Let $a, b \in H$, $b^{-1} \in H$ $(\because$ of existence of inverse)

$\Rightarrow ab^{-1} \in H \; \forall \, a, b^{-1} \in H$ $(\because$ of closure property)

Conversely : $\forall \, a, b \in H \Rightarrow ab^{-1} \in H$ T.P. H is a group in itself.

Now $a, a \in H \Rightarrow aa^{-1} \in H$ $(\because$ of given$) \Rightarrow e \in H$

Further $e, a \in H \Rightarrow ea^{-1} \in H$ $(\because$ of given$) \Rightarrow a^{-1} \in H$

Let $a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow a\,(b^{-1})^{-1} \in H \Rightarrow ab \in H$ $(\because (b^{-1})^{-1} = b)$

Also associative law holds for large set. $\therefore$ It must holds for smaller set H. $\therefore$ H is a group in itself.

Hence, H is a subgroup of group G.

Q 34. Is the set Z of integers with the binary operation of subtraction a semi-group. Justify your answer. **(PTU, May 2013)**

Solution. $\forall \, a, b \in I \Rightarrow a - b \in I$ (As subtraction of two integers is also an integer)

$\therefore$ I is closed under subtraction.

$\forall \, a, b, c \in I$

$$a - (b - c) = (a - b) - c$$

Thus, Z is a semi-group under binary operation subtraction. [Associative law holds]

Q 35. Give an example of a non-abelian group of order 8. **(PTU, May 2013)**

Solution. Non-abelian group :

Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ define product on G by usual multiplication together with $i^2 = j^2 = k^2 = -1$; $ij = -ji = k$, $jk = -kj = i$ and $-ik = ki = j$

Composition table

	+1	−1	i	−i	j	j	k	−k
+1	1	−1	i	−i	j	−j	k	−k
−1	−1	1	−i	i	−j	+j	−k	k
i	i	−i	−1	1	k	−k	−j	j
−i	−i	i	1	−1	−k	k	+j	−j
j	j	−j	−k	k	−1	+1	i	−i
−j	−j	j	k	−k	+1	−1	−i	i
k	k	−k	j	−j	−i	i	−1	+1
−k	−k	k	−j	j	i	−i	+1	−1

Clearly G is closed under multiplication, Associative law holds 1 be the identity.

The inverses of $\pm 1, \pm i, \pm j, \pm k$ are $\pm 1, \mp i, \mp j, \mp k$.

Clearly $ij \neq ji$ $\therefore$ G is non-commutative *i.e.* non-abelion of order 8.

Q 36. Let $\angle 1$ be the additive group of integers. Prove that map $f : \angle 1 \to \angle 1$ defined by $f(x) = 2x$, $x \in \angle 1$ is a group isomorphism. **(PTU, Dec. 2012)**

Solution. $f : \angle 1 \to \angle 1$ defined by $f(x) = 2x \; \forall \, x \in \angle 1$

$\forall \, x, y \in \angle 1$

$$f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$$

$\therefore$ f is homomorphism on $\angle 1$.

1 – 1 : $\forall \, x, y \in \angle 1$ s.t $f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$

$\therefore$ f is 1 – 1.

Thus f is 1 – 1 iand homo. $\therefore$ f is group isomorphism.

□□□

Chapter

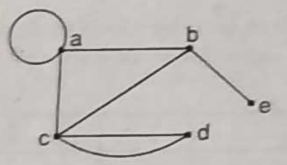
8

Trees & Graph Theory

QUESTION-ANSWERS

Q 1. Define multigraph. **(PTU, Dec. 200...)**

Solution. Multigraph : A graph (V, E) having either loops or multiple edges or both is called so. e.g : As there are two edges between c and d and there is a loop at vertex a.



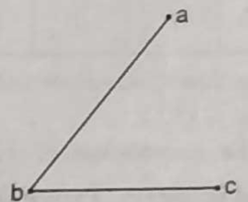
Q 2. What is indegree and outdegree of a graph? **(PTU, Dec. 2006, 200...)**

OR

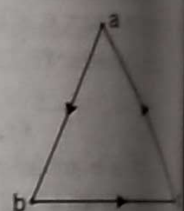
How you define degree in a graph? **(PTU, Dec. 2006 ; May 20...)**

Solution. Degree of vertex : Degree of a vertex v in a graph G is the number of edges incident on v, written as deg (v).

e.g. degree of each vertex in Fig. (i) deg (a) = 1, deg (b) = 2, deg (c) = 1.



If G is a directed graph, degree of a vertex is the sum of outdegree and indegree of that vertex. Outdegree of a vertex v is the number of edges whose initial vertex is v and indegree of a vertex v is the number of edges which have terminal vertex v. A vertex v with zero indegree is called a source and a vertex v with zero outdegree is called a sink.



e.g : Here indegree of a, b, c are 1, 1, 1 and outdegrees of a, b, c are 1, 1, 1.

Q 3. Define the closed path and a cycle in graph. **(PTU, May 2003)**

Solution. Path : Given a multigraph G and there is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots\ldots v_{i-1}, e_i, v_i, \ldots e_n, v_n\}$

Such a sequence is called 'Path'. We can also write this sequence as $\{v_0, v_1, v_2, \ldots v_{n-1}, v_n\}$.

Further, Number of edges in path is called 'length' of path. When all vertices are distinct, path is called **'Simple Path'**. If in a path, all edges are distinct, then it is called a **'trial'**.

Cycle : A closed path is called a cycle, if all vertices are distinct except $v_0 = v_n$.

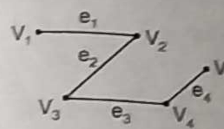


Fig. (i)

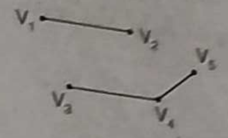


Fig. (ii)

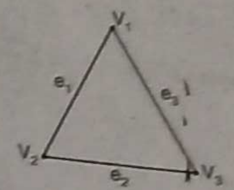


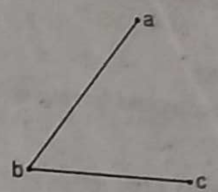
Fig. (iii)

Fig. (i) is an example of a path $\{v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5\}$ and Fig. (ii) is not a path as there is no edge between v_2 and v_5 etc. fig. (iii) is a cycle given by $\{v_1, e_1, v_2, e_2, v_3, e_3, v_1\}$

Q 4. Differentiate between a directed and unindirected graph. **(PTU, Dec. 2010, 2003)**

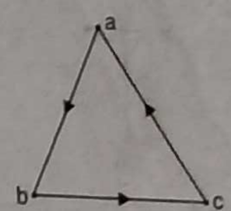
Solution. A graph is consisting of two sets : a set of vertices V and a set of edges E obtained by joining certains vertices of V. And it is denoted by (V, E).

Example 1 : If V = {a, b, c}, E = {(a, b), (b, c)} then graph (V, E) is as follows ; where order of vertices, does not matter in E.



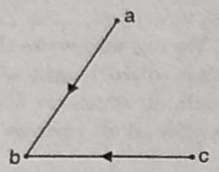
Undirected graph

Example 2 : Given V = {a, b, c} E = {(a, b), (b, c), (c, a)} and (V, E) is as under, order of vertices in E matters.



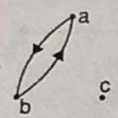
Directed graph

Example 3 : Given V = {a, b, c}, E = {(a, a), (a, b), (b, c)} then (V, E) is



Directed graph

Example : If V = {a, b, c}, E = {(a, b), (b, a)} and graph (V, E) is as



Directed and Undirected graph

Directed graph is a graph in which all edges have some direction. In such graphs if edge e = (a, b) an ordered-pair. Vertex a is the initial vertex and vertex b is the terminal vertex of the edge (a, b) ≠ (b, a). Otherwise graph is called undirected and in undirected graph (a, b) = (b, a).

Q 5. Define a simple path and a trial in a graph. **(PTU, Dec. 2003)**

Solution. Number of edges in path is called 'length' of path. When all vertices are distinct, path is called **'Simple Path'.** If in a path, all edges are distinct, then it is called a **'trial'.**

Trivial graph : A graph having one vertex and having no edge is called so.

e.g ⊙ v is a trivial graph.

Q 6. What is the trivial graph? **(PTU, Dec. 2003 ; May 2003)**

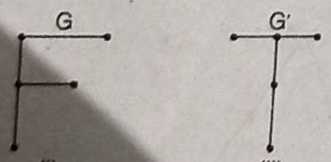
Solution. A graph having one vertex and having no edge is called so.

e.g ⊙ v is a trivial graph.

Q 7. Define isomorphic graph. **(PTU, Dec. 2004, 2003)**

Solution. Two graphs G (V, E) and G' (V', E') are said to be isomporhic if there exists one-to-one correspondence $f : V \to V'$ such that (u, v) is an edge of G iff [f (u), f (v)] is an edge of G'.

For example :



Number of edges in both graphs = 4

degree sequence in both graphs = (1, 2, 3, 1, 1)

Degree seq. of each neighbourhood vertex of every vertex = (2), (1, 3), (2, 1, 1), (3), (3) For (i) and (2), (1, 3), (2, 1, 1), (3), (3) For (ii).

∴ There is one-to-one correspondence between G and G'.

So, two graphs are isomorphic.

Q 8. Define the following concepts from graph theory with an example for each:

(i) Spanning subgroup of a graph.

(ii) A full regular binary tree **(PTU, Dec. 2004)**

Solution. (i) Spanning Tree : Let G be a connected graph and T be a subgroup of G. Then T will be called spanning tree of G if T is a tree and contains all the vertices of G.

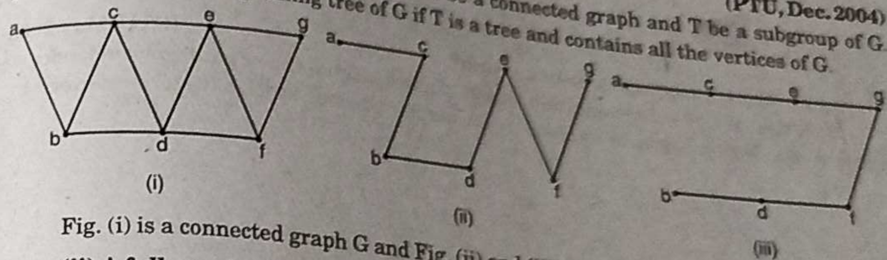
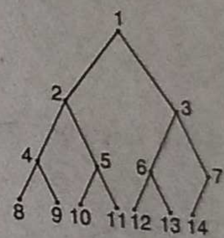


Fig. (i) is a connected graph G and Fig. (ii) and (iii) are spanning trees of G.

(ii) A full regular binary tree : If in a binary tree T, each parent vertex has exactly two children except possible the last vertex then T is complete binary tree.

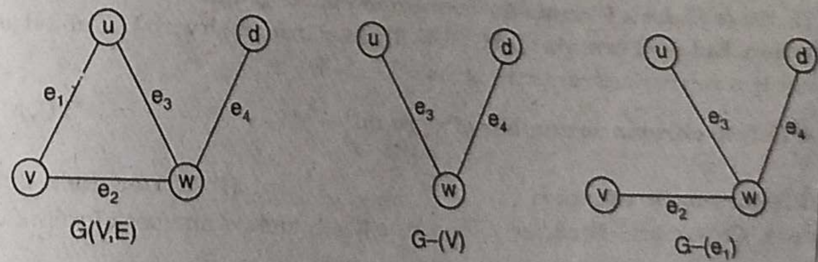


Q 9. What is a subgraph? **(PTU, Dec. 2006 ; May 2**

Solution. Let G (V, E) be any graph then S (V', E') will be called the subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

e.g. G – {v} is subgraph of G obtained by deleting the vertex v and edges incident

And G – {e_1} is the subgraph of G obtained by deleting the edge e_1 only.

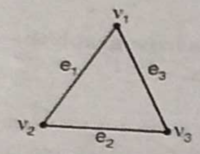


is ....

c. 2009)

Q 10. What is a cycle in a graph? **(PTU, May 2005)**

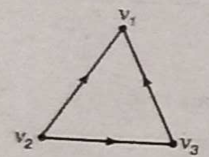
Solution. A closed path is called a cycle if all its vertices are distinct except $v_0 = v_n$ i.e. cycle in a graph is a circuit whose edge list does not contain repitition e.g. $\{v_1, e_1, v_2, e_2, v_3, e_3, v_1\}$ forms a cycle.



Q 11. What is the indegree of a graph? **(PTU, May 2005)**

Solution. If G be a directed graph, degree of a vertex is the sum of outdegree and indegree of that vertex. Now indegree of a vertex v is the number of edges which have terminal vertex v.

e.g.



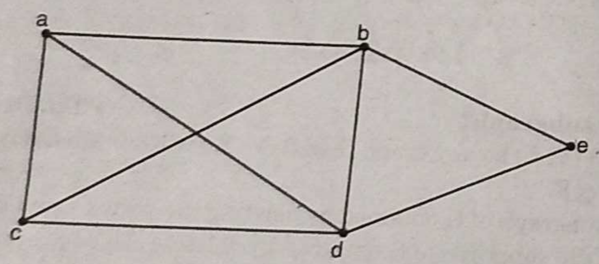
Here indegree of $v_1 = 2$

Indegree of $v_2 = 0$

Indegree of $v_3 = 1$.

Q 12. What is connected graph? **(PTU, Dec. 2005)**

Solution. A graph G is said to be connected if every two vertices of that graph have a path between them.



Q 13. State Euler's formula for connected planar graph. **(PTU, May 2006)**

Solution. Euler's Formula : Let G (V, E, R) be a planar graph with V, E and R as number of vertices, edges and regions respectively then $V - E + R = 2$.

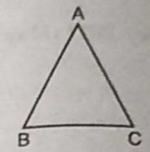
Q 14. Define chromatic number of a graph. **(PTU, May 2006)**

OR

What is chromatic number? **(PTU, Dec. 2009, 2005, 2004)**

Solution. Chromatic Number χ (G) : It is the minimum number of colors to color a graph.

For example :



; χ (G) = 3

Q 15. Differentiate between paths and circuits. **(PTU, May 2007)**

Solution. Path : Given a multigraph G and there is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots\ldots\, v_{i-1}, e_i, v_i, \ldots\, e_n, v_n\}$

Such a sequence is called 'Path'. We can also write this sequence as $\{v_0, v_1, v_2, \ldots v_{n-1}, v_n\}$.

Note : Number of edges in path is called 'length' of path. When all vertices are distinct, path is called **'Simple Path'**. If in a path, all edges are distinct, then it is called a **'trial'**.

Cycle : A closed path is called a cycle, if all vertices are distinct except $v_0 = v_n$.

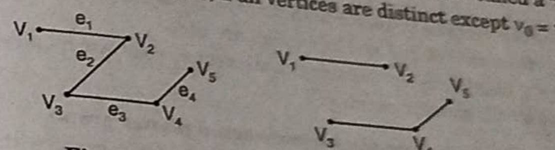


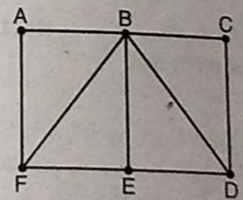
Fig. (i) **Fig. (ii)**

Fig. (i) is an example of a path $\{v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5\}$ and Fig. (ii) is not a path as there is no edge between v_1 and v_5 etc.

Q 16. Define Hamiltonian cycle. **(PTU, Dec. 2007)**

Solution. Hamiltonian Cycle : It is a circuit through graph where vertex list contains each vertex of the graph exactly once except the initial vertex appears as second time as the terminal vertex.

e.g.



It contains Hamiltonian cycle ABCDEFA.

Q 17. Postfix expression for the infix expression A + B * (C + D)/F + D * E is .... **(PTU, Dec. 2009)**

Solution. Postfix expression is given by

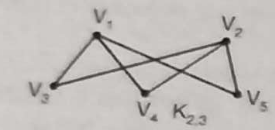
AB + CD + F / *DE* +

OR (PTU, Dec. 20..)

What is the chromatic number of $K_{2,3}$? (PTU, Dec. 2011 ; May 2010)

Solution.

Given vertices	V_1	V_2	V_3	V_4	V_5
decreasing order of degrees	3	3	2	2	2
Colors	a	a	b	b	b



We color the vertices in such a manner that, no two adjacent vertices having same color.

∴ all vertices are colored and two colors are used.

∴ $\chi(G) \le 2$ also V_1, V_3 are adjacent ∴ atleast two color are required.

∴ $\chi(G) = 2$ = Chromatic number.

Q 19. State Euler's formula for connected planar graph. (PTU, Dec. 2007)

Solution. Euler's Formula : Let G (V, E, R) be a planar graph with V, E and R as number of vertices, edges and regions respectively then $V - E + R = 2$.

Q 20. How many edges are there in a graph with 10 vertices each of degree six? (PTU, May 2010)

Solution. We know that, sum of degrees of all vertices of graph G is equal to twice the number of edges in G.

i.e. $\Sigma \deg G(v_i) = 2\,|E|$

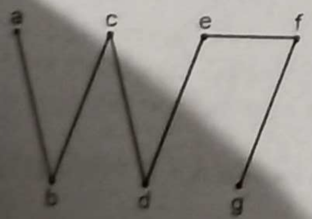
where $|E|$ = No. of edges in G.

i.e. $10 \times 6 = 2\,|E| \Rightarrow |E| = 30$.

Q 21. Define a tree. (PTU, May 2012)

Solution. Tree is a graph which is connected and has no cycles.

e.g.



If graph G be a tree with n vertices then it has n – 1 edges.

Q 22. Write short notes on the following : Eulerian path and circuits.

OR

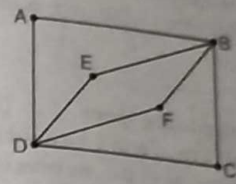
Define following terms with examples. Euler graph. (PTU, May 2010 ; Dec. 2007, 2006)

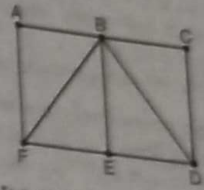
(PTU, Dec. 2009 ; May 2011, 2009, 2004)

Solution. Eulerian Path : It is a path through a graph in which no edge is repeated twice. It means, edge list contains each edge exactly once.

Eulerian Circuit : If Eulerian path forms a circuit, then it is known as Eulerian circuit.

Example 1.

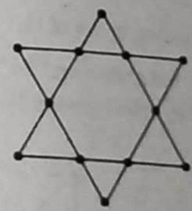


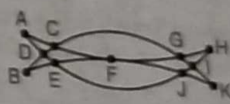


(i) Here every vertex is of even degree ∴ It is Eulerian. It is non-Hamiltonian as there is no path which contains vertices exactly once.

(ii) It is Hamiltonian as it contains Hamiltonian cycle ABCDEFA. It is not Eulerian as it has vertices B, D, E, F which has odd degree. So more than two vertices is of odd degree. ∴ There can be no Euler path.

Example 2.





(i) It is an Eulerian circuit as every vertex is of even degree.

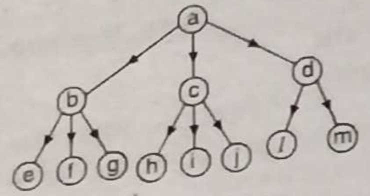
(ii) It is again Eulerian as each vertex is of even degree.

Q 23. Define a rooted tree T with an example and show how it may be viewed as directed graph. (PTU, Dec. 2010, 2002)

Solution. First of all, we define directed tree, it is a directed graph which is a tree when the directions of the edges are ignored.

Rooted tree is a directed tree which has exactly one node called root has i... which all other nodes having indegree 1. The vertex which has outdegree 0 is c... external node or leaf and the vertices which having outdegree greater than 1 or internal nodes.

e.g.



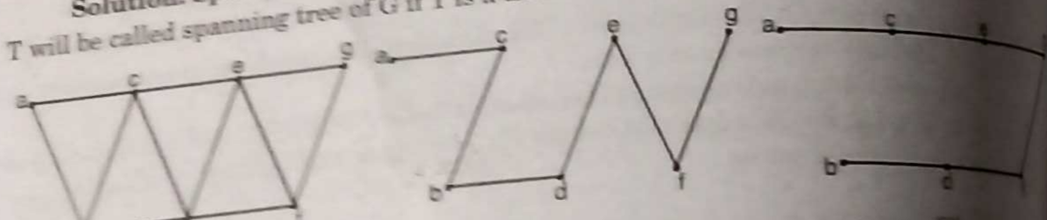
Rooted tree

Now we know that, tree is a connected graph and without cycles and rooted tree T is a tree which having a designated vertex called root of the tree and there is a unique and simple path from root to every vertex of T.

So T may be considered as a directed graph.

Q 24. Mention the properties of minimum spanning trees. **(PTU, Dec. 200**

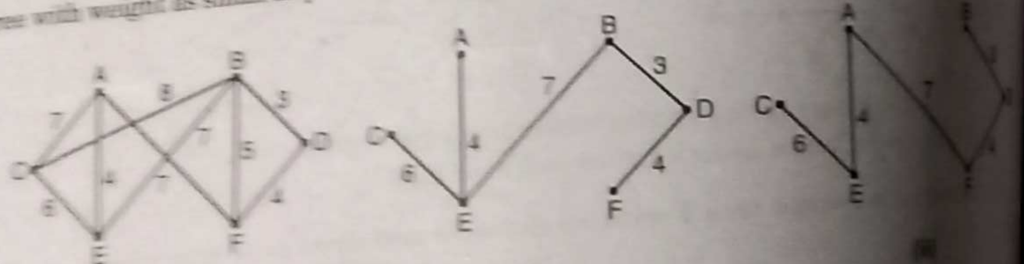
Solution. Spanning Tree : Let G be a connected graph and T be a subgraph of G. The T will be called spanning tree of G if T is a tree and contains all the vertices of G.



(i) (ii) (iii)

Fig. (i) is a connected graph G and fig. (ii), (iii) are spanning trees of G.

Minimum Spanning Tree : Let G be a connected weighted graph and let T be its spanning tree with weight as small as possible. (weight of T is obtained by adding weight of all edges of



(i) (ii) (iii)

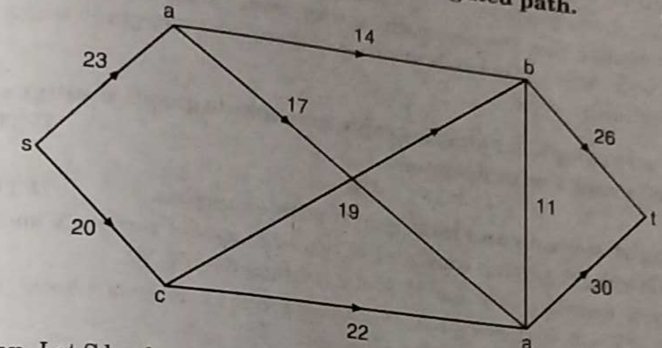
Fig. (i) is graph G and Fig. (ii) and (iii) are min. spanning trees with weight
graph G has 6 vertices, therefore, min. spanning tree will have 5 edges.

Properties of minimum spanning tree :

1. The total weight of the spanning tree is the sum of the weights of the
spanning trees.

2. The minimum weight of the spanning tree is unique.

Q 25. Using either Breath First search Algorithm or Dijkstra's algorithm, find the shortest path from s to t in the following weighted path. **(PTU, Dec. 2004)**



Solution. Let S be the set of vertices which are included to find shortest path. Start with S including source vertex s.

(i) **S** — **Distance to all other vertices**

S	s	a	b	c	d	t
's'	0	23 (s)	∞	20 (s)	∞	∞

Here vertex c is nearest to 's' so c must be included in S and find shortest path to all vertices through this vertex and update the distances.

(ii)

S	s	a	b	c	d	t
's, c'	0	23 (s)	39 (s, c)	20 (s)	42 (s, c)	∞

Here a is nearest to s

(iii)

S	s	a	b	c	d	t
's, c, a'	0	23 (s)	37 {s, a}	20 (s)	40 {s, a}	63 {s, a, b}

Here b is nearest to s

(iv)

S	s	a	b	c	d	t
's, c, a, b'	0	23 (s)	37 (s, a)	20 (s)	40 {s, a}	63 {s, a, b}

Here d is nearest to s

(v)

S	s	a	b	c	d	t
s, c, a, b, d	0	23 (s)	37 (s, a)	20 (s)	40 {s, a}	63 {s, a, b}

Since S includes n – 1 i.e. 6 – 1 = 5 vertices ∴ process ends.

Hence the shortest path is {s, a. b, t} and is of length 63.

Q 26. Suppose a graph G contains two distinct paths from a vertex a to b. Show that G has a cycle. **(PTU, May 2003)**

Solution. Let the graph G is having two different paths given by

$$P_1 = \{e_1, e_2, \ldots e_n\} \text{ and } P_2 = \{e'_1, e'_2, \ldots e'_m\}$$

starting from vertex a to vertex b. Now let us delete those edges from two paths P_1 and P_2 which are identical i.e. if $e_1 = e'_1$; $e_2 = e'_2$; ..... $e_i = e'_i$ but $e_{i+1} \neq e'_{i+1}$ so we delete $(e_1, ... e_i)$ and $(e'_1, ... e'_i)$ from P_1 and P_2.

Now let us assume that the new path starting from vertex a', let us construct a cycle which is starting with vertex a' and ends at vertex d' therefore every graph which contains two distinct paths must have a cycle.

Q 27. Give an example for simple graph, non-simple graph, multigraph, directed graph, weighted graphs with diagrams. **(PTU, Dec. 2006)**

OR

Define weighted graph and multigraph with examples. **(PTU, May 2007)**

Solution. Graph : A graph is consisting of two sets : a set of vertices V and a set of edges E obtained by joining certains vertices of V. And it is denoted by (V, E).

Example 1. *If V = {a, b, c}, E = {(a, b), (b, c)} then graph (V, E) is as follows ; where order of vertices, does not matter in E.*

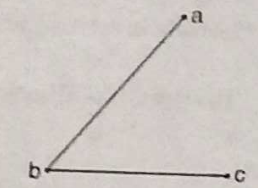


Fig. (i)

Example 2. *If V = {v_1, v_2}, E = ϕ then graph (V, E) will be as under : order, does not matter in E.*

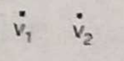


Fig. (ii)

Example 3. *Given V = {a, b, c}, E = {(a, b), (b, c), (c, a)} and (V, E) is as under, order of vertices in E matters.*

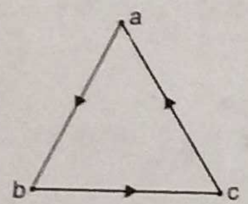


Fig. (iii)

Example 4. *Given V = {a, b, c}, E = {(a, a), (a, b), (b, c)} then (V, E) is*

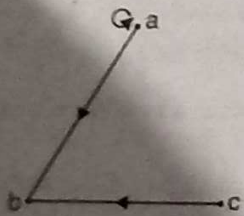


Fig. (iv)

Example 5. *If V = {a, b, c}, E = {(a, b), (b, a)} and graph (V, E) is as*

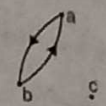


Fig. (v)

Multigraph : A graph (V, E) having either loop or multiple edges or both is called so.

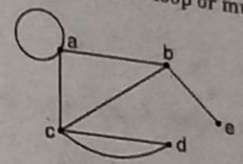


Fig. (vi)

So, graphs without loops or multiple edges are known as 'Simple graphs'. Fig. (i), (ii), (iii), (v) while (iv) and (vi) forms a multigraph.

Directed Graph : It is a graph in which all edges have same direction. In such graphs if edge e = (a, b) an ordered-pair. Vertex a is the initial vertex and vertexb is the terminal vertex of the edge (a, b) ≠ (b, a). Fig. (iii), (iv) and (v).

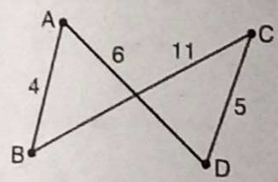
Otherwise graph is called undirected and in undirected graph (a, b) = (b, a).

Labelled and Weight Graphs : A graph G is called a labelled graph if its edges and/or vertices are assigned data of any kind.

A graph is called **Weighted graph** if each of its edge has been assigned some non-negative value w (e) called the weight or length of edge e.

Weight or length of the path is the sum of the weights or lengths of edges of the path.

For example :



In this weighted graph length of path = 4 + 6 + 5 + 11 = 26.

Q 28. Let T be a binary tree with n vertices. Determine the number of leaf nodes in tree. **(PTU, Dec. 2008)**

Solution. Since T is a binary tree with n-vertices

∴ each parent vertex has atmost two children.

In a tree, we have three types of nodes

Let n_0 = no. of nodes having degree zero

n_1 = no. of nodes having degree one

n_2 = no. of nodes having degree two.

$\therefore \quad n = n_0 + n_1 + n_2$

Let the tree T having e edges, now these e-edges connects (e + 1) nodes. ...(1)

Thus $\quad n = e + 1$

Further all these edges are coming either from nodes of degree 1 or degree 2 ...(2)

$\therefore \quad e = n_1 + 2n_2$

From (2) and (3) ; we have ...(3)

$n = n_1 + 2n_2 + 1$ ...(4)

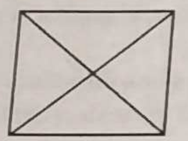
From (1) and (4) ; we have

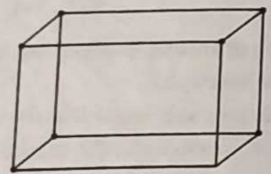
$n = n_1 + 2(n - n_0 - n_1) + 1 \Rightarrow n = n_1 - 2n_0 + 1$

$\therefore$ number of leafs in a binary tree

$= \text{nodes of degree one} = n_1 = n + 2n_0 - 1.$

Q 29. Show that following graphs are planar. **(PTU, May 2009)**





Solution.

(a) Here $|V| = 5$; $|E| = 8$

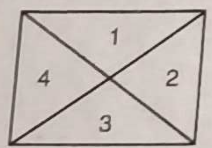
and $|R| = 5$

Now $|V| - |E| + |R| = 5 - 8 + 5 = 2$

$\therefore$ Euler's formula satisfied.

Further for a planar graph $|E| \le 3|V| - 6$

i.e. $8 \le 15 - 6 = 9$ which is true.



Also for a planar graph, we have $2|E| \ge 3|R|$

i.e. $2 \times 8 \ge 3.5$ which is true.

Thus given graph is a planar graph.

(b) Here, $|V| = 8$; $|E| = 12$; $|R| = 6$

Now, $|V| - |E| + |R| = 8 - 12 + 6 = 2$ $\therefore$ Euler's formula satisfied.

Futher for a planar graph $|E| \le 3|V| - 6$

i.e. $12 \le 3.8 - 6 = 18$ which is true.

Also, for planar graph $2|E| \ge 3|R|$

i.e. $2.12 \ge 3.6$ which is true.

Hence graph is a planar graph.

Q 30. Suppose a directed graph G has m vertices. Show that if there is a path P from vertex u to v, then there is a path p of length m–1 or less from u to v. **(PTU, May 2003)**

Solution. We know that, simple path in a directed graph is one in which there are no repeated vertices within the path, also the number of distinct vertices in a simple path of length n is n + 1 because the length of the path is the number of edges appearing in the sequence of path. As there are only m distinct vertices in the graph so we can't have a simple path of length greater than m – 1 hence a graph having m vertices have a path of length m – 1 or less.

Q 31. What is an Eulerian circuit? Prove that an undirected graph G possesses an Eulerian circuit if and only if it is connected and its vertices are all of even degree. **(PTU, Dec. 2003)**

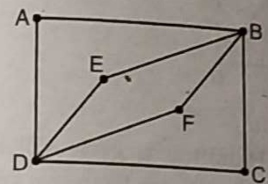
OR

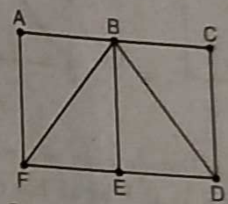
Prove that an undirected graph G possesses an Eulerian circuit iff it is connected and all its vertices are of even degree. **(PTU, May 2003)**

Solution. Eulerian Path : It is a path through a graph in which no edge is repeated twice. It means, edge list contains each edge exactly once.

Eulerian Circuit : If Eulerian path forms a circuit, then it is known as Eulerian circuit.

Example 1.

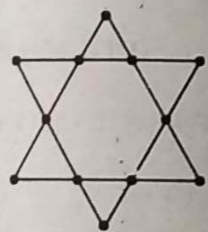




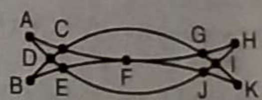
(i) Here every vertex is of even degree $\therefore$ It is Eulerian. It is non-Hamiltonian as there is no path which contains vertices exactly once.

(ii) It is Hamiltonian as it contains Hamiltonian cycle ABCDEFA. It is not Eulerian as it has vertices B, D, E, F which has odd degree. So more than two vertices is of odd degree. $\therefore$ There can be no Euler path.

Example 2.



(i)



(ii)

(i) It is an Eulerian circuit as every vertex is of even degree.

(ii) It is again Eulerian as each vertex is of even degree.

For a connected multigraph G, we want to prove that G is Eulerian iff every point of G has even degree.

Proof : Let P be the Eulerian path in G. Now every vertex of G contributes two to the degree. Since G is Eulerian. ∴ Every edge of G appears exactly once in G. ∴ every vertex of G has even degree.

Converse : Further let the undirected graph is connected and all vertices are of even degree. We want to prove that graph possess Eulerian circuit. No edge will be traced more than once as graph is connected and all vertices is of even degree. Whenever the circuit enters the vertex (node) through an edge leave the node through another edge which has not been traced before and the initial and terminal vertex of the circuit must be same.

Q 32. Draw graph which is :

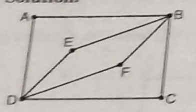
(i) Hamiltonian and Non Eulerian.

(ii) Non-Hamiltonian and Eulerian. **(PTU, May 2004)**

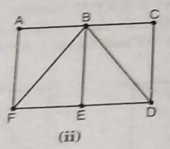
OR

Give an example of a graph which is Hamiltonian but not Eulerian and vice versa. **(PTU, Dec. 2010)**

Solution.



(i)



(ii)

(i) In fig. (ii), Graph is Hamiltonian as it contains Hamiltonian cycle ABCDEFA. It is not Eulerian as it has vertices B, D, E, F which has odd degree. So more than two vertices is of odd degree. ∴ There can be no Euler path.

(ii) In fig. (i), Here every vertex is of even degree. ∴ It is Eulerian. It is non-Hamiltonian as there is no path which contains vertices exactly once.

Q 33. In each of the following expressions, what is the coefficient in front of the term whose exponents is 4?

(i) $(1 + x + x^2 + x^3 + x^4)^3$

(ii) $(1 + x^2 + x^4)^2 (1 + x + x^2)$

(iii) $(1 + x + x^2 + x^3 + x^4 + ...)^3$. **(PTU, May 2004)**

Solution. (i) $(1 + x + x^2 + x^3 + x^4)^3 = [(1 + x)(1 + x^2) + x^4]^3$

$$= (1 + x)^3 (1 + x^2)^3 + x^{12} + 3x^4 (1 + x)^2 (1 + x^2)^2 + 3x^8 (1 + x)(1 + x^2)$$

$$= [1 + x^3 + 3x^2 + 3x][1 + x^6 + 3x^2 + 3x^4] + x^{12}$$

$$+ 3x^4 (1 + x^2 + 2x)(1 + x^4 + 2x^2) + 3x^8 (1 + x)(1 + x^2)$$

∴ Required coefficient of $x^4 = 3 + 9 + 3 = 15$

(ii) $(1 + x^2 + x^4)^2 (1 + x + x^2) = [1 + x^4 + x^8 + 2x^2 + 2x^6 + 2x^4][1 + x + x^2]$

∴ Required coefficient of $x^4 = 1 + 2 + 2 = 5$

(iii) $(1 + x + x^2 + x^3 + x^4 + ...)^3 = [(1 - x)^{-1}]^3 = (1 - x)^{-3}$

$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + ...$$

∴ Coefficient of $x^4 = 15$.

Q 34. Prove that an undirected graph has an Euler path if and only if it is connected and has '0' or exactly '2' vertices of odd degree. Give example. **(PTU, Dec. 2004)**

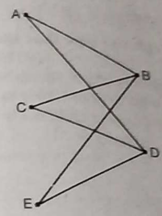
OR

If an undirected graph which is connected and has zero or exactly 2 vertices of odd degree. Show that it has an Euler path. **(PTU, Dec. 2003)**

Solution. Let the undirected graph has an Euler path so by definition, it is a connected graph as graph is Eulerian therefore its edge list contains each edge exactly once so it means, path in every time, meets a node and path goes through two edges which are incident with a vertex, thus degree of all other vertices except the two vertices at the end of the path is of even degree, further if there two distinct nodes at the end of Eulerian path then there are only two vertices with odd degree.

Converse : Let u and v are two vertices which are of odd degree in G by adding the edge (u, v) to G we can produce a connected graph G' (say) all of whose vertices are of even degree we can get a Euler circuit in G'. Deleting the edge (u, v) from the circuit we can get a Euler path that begins at (u or v) and ends at (v or u).

e.g.



Here vertex B and D is of odd degree i.e. 3 and A, C, E having even degree i.e. 2

∴ This graph has a Eulerian path.

Q 35. What is a planar graph? Prove that V – E + R = 2. **(PTU, 2004)**

OR

State and prove Euler's formula in connected maps.

(PTU, Dec. 2011, 2006 ; May 2006)

OR

Consider any connected planar graph G = (V, E) having R regions, V vertices and E edges. Show that V + R – E = 2. **(PTU, May 2012, 2011, 2010)**

Solution. Planar Graphs : A graph or multigraph is said to be a planar graph if its edges do not cross while drawing it on a plane. For example, the complete graph k_3 is planer.

Planar representation of a finite planer multigraph is called a **map.** A map divides the plane into various sections called **regions.** Regions may be bounded or unbounded. Degree of the region, deg r, is the length of the cycle which makes the boundary of that region.

Euler's Formula : Let G (V, E, R) be a planar graph with V, E and R as number of vertices, edges and regions respectively then V – E + R = 2.

Proof : We shall prove the result by induction on R.

Let $R = 1 \Rightarrow E = 0$ and $V = 1 \therefore V - E + R = 1 - 0 + 1 = 2$

$\therefore$ The result is true for R = 1.

Let us assume the result for $R = m, m \in N$

$\therefore$ We want to prove the result for $R = m + 1$.

On removing an edge now which will be common to the boundary of two regions, we obtain a new graph G' (V', E', R') which has m regions now.

For m regions, the result is true. $\therefore V' - E' + R' = 2$ ...... (1)

Since, we have obtained G' by removing an edge.

$\therefore$ Number of vertices remain same.

$\because$ $V = V'$ and $E' = E - 1$ and $R' = R - 1$

(as on removing common edge between two regions, we get one region only) ...... (2)

$\therefore$ for m + 1 regions,

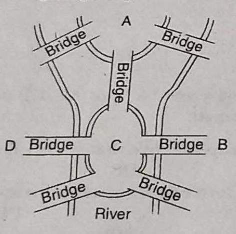
$$V - E + R = V' - (E' + 1) + (R' + 1) \quad \text{[using (2)]}$$
$$= (V' - E' + R') - 1 + 1 = 2 \quad \text{[using (1)]}$$

Hence, the result is true for $R = m + 1$ therefore by induction $V - E + R = 2$.

Q 36. State and prove Eulerian theorem on graph to show that Koninsberg's graph is not proved as a solution. **(PTU, Dec. 2004)**

Solution. The Konigsberg Bridge Problem : A Prussian city of Konigsberg has four land masses and seven bridges. Many citizens of this city tried to plan a tour to pass seven bridges without crossing any of them more than once. But such a tour could not be designed.

Following is the map of Konigsberg bridge :



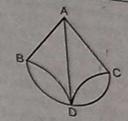
(i)

Four land masses A, B, C and D cannot be covered by crossing seven bridges, eac of them exactly once.

In this case, 'Euler' presented the following result :

...king tour of Konigsberg can be possible to cross each bridge exactly once'.

As a proof, he says that above map can be considered as a multigraph with four mass lands ...ces and seven briges as edges, see fig. (ii). Since path can start and end at two different vertices therefore remaining two vertices will be intermediate vertices. To visit these vertices, each vertex is used twice, that is, two edges are connected with it one to enter and other to exit.



So, even number of edges are required to connect there vertices with the others. In Konigsberg graph, each vertex has odd degree, so no tour of the said type is possible.

Q 37. What are the applications of graph theory in computer science? Explain with example. **(PTU, Dec. 2005)**

Solution. Graph theory plays a very useful role in computer science and providing the solution to two most important problems *i.e.* the Travelling Salesman Problem and the Maximum Flow Problem.

1. Travelling salesman problem : For explaining such problem first of all, we define weighted graph.

A weighted graph, (V, E, w) is a graph (V, E) together with a weight function $w : E \rightarrow R$. If $e \in E$, w (e) is the weight on edge e.

The Travelling Salesman Problem

Given a weighted graph, find a circuit $(e_1, e_2, ..., e_n)$ that visits every vertex at least once and minimizes the sum of the weights, $\sum_{i=1}^{n} w(e_i)$

Any such circuit is called an optimal path and is a solution to the Travelling Salesman Problem.

OR

In other words, suppose a sales representatives based on city A wishes to visit cities B,C,D,E & F exactly once and return home. He also wants to minimise the distance and this problem ask for Hamiltonian circuit of travelled minimum length. Here we represents vertices of weighted graph as cities and the edge weights as distances the solution to travelling salesman problem is to find shortest path in which salesman can visit each city exactly once and return back to his city.

For example : A robot is programmed to weld joints on square metal plates. Each plate must be welded at prescribed points on the square. To minimize the time it takes to complete the job, the total distance that the robot's arm moves should be minimized. Let d (P, Q) be the distance between P and Q. Assume that before each plate can be welded, the arm must be positioned at a certain point P_0. Given a list of n points, we want to put them in order so that

$$d(P_0, P_1) + d(P_1, P_2) + ... + d(P_n, P_0)$$

is as small as possible.

The graph theory helps to construct a graph. K_n and find a circuit of the graph that minimizes the sum of the distances traveled in traversing the circuit.

2. **Networks and the maximum flow problem :** A network is a simple weighted directed graph that contains two distinguished vertices called the source and sink. An example of a real situation that can be represented by a network is a city's water system. A reservoir would be the source, while a distribution point in the city of all of the users would be the sink. The system of pumps and pipes that carriers the water from source to sink makes up the remaining network. We can assume that the water that passes through a pipe in one minute is controlled by a pump and the maximum rate is determined by the size of the pipe and the strength of the pump. This maximum rate of flow through a pipe is called its capacity and is the information that the weight function of network contains. Now by using graph theory we want to maximise the rate of flow through the pipes.

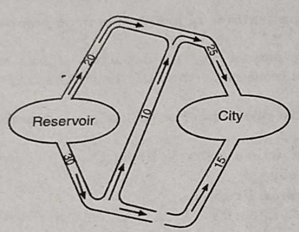


Fig. 1

Let us Consider the system as shown in Fig. 1. The numbers that appear next to each pipe indicate the capacity of that pipe in thousands of gallons per minute. This map can be drawn in the form of a network, *i.e.* is a graph given below by fig. 2 and it is also the solution of our problem.

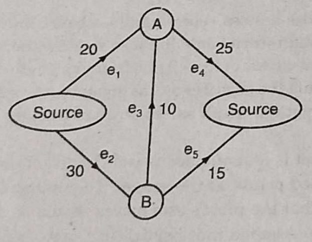


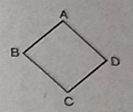
Fig. 2

Q 38. Write short notes on the following :

Hamiltonian Graphs. **(PTU, May 2007 ; Dec. 2005)**

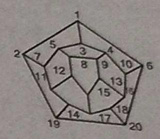
Solution. **Hamiltonian Graph :** A graph that possesses Hamiltonian Path is said to be Hamiltonian graph.

'Dirac condition for Hamiltonian graph' : Let G be a connected graph having n vertices. Then G will be Hamiltonian if $n \geq 3$ and $n \leq \deg(V)$ for each vertex V in G.



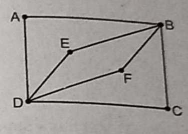
Eulerian Circuit

∴ degree of each vertex is even.

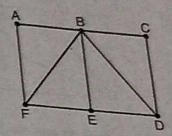


Hamiltonian Graph

e.g.



(i)



(ii)

(i) Here every vertex is of even degree ∴ It is Eulerian. It is non-Hamiltonian as there is no path which contains vertices exactly once.

(ii) It is Hamiltonian as it contains Hamiltonian cycle ABCDEFA. It is not Eulerian as it has vertices B, D, E, F which has odd degree. So more than two vertices is of odd degree. ∴ There can be no Euler path.

Q 39. Give an example of a graph and explain for the following :

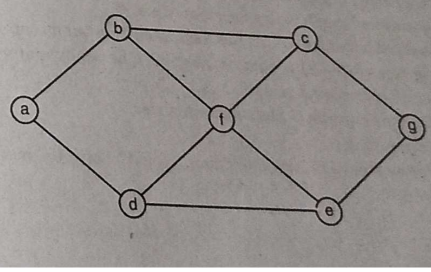
(a) A graph is having Hamiltonian and Euler Circuit.

(b) A graph is having Hamiltonian Circuit but not an Euler Circuit.

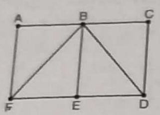
(c) A graph is having Euler Circuit but not an Hamiltonian circuit.

(PTU, May 2010 ; Dec. 2009)

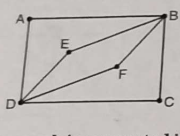
Solution. (a) It is both Eulerian and hamiltonian.



(b) It is Hamiltonian as it contains Hamiltonian cycle ABCDEFA. It is not Eulerian as it has vertices B, D, E, F which has odd degree. So more than two vertices is of odd degree ∴ There can be no Euler path.



(c) Here every vertex is of even degree ∴ It is Eulerian. It is non-Hamiltonian as there is no path which contains vertices exactly once.



Q 40. Prove that a simple graph is connected if and only if it has a spanning tree.

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(PTU, Dec. 2008)

Solution. Let G be a connected graph (given)

To prove : G has a spanning tree

Let us suppose that G has r-cycles.

If r = 0 then G has no cycle also it is connected ∴ G is a tree.

Now let us suppose that all connected graph with less than r-cycles have a spanning tree. Since G be a connected graph with r-cycles and let e be the edge in one of its cycles.

∴ G – {e} is a connected graph with edges fewer than G.

But by mathematical induction, G – {e} has a spanning tree but it contains all vertices of G.

∴ The spanning tree of G – {e} is same as that of G, hence by mathematical induction, result is holds good for all connnected graphs.

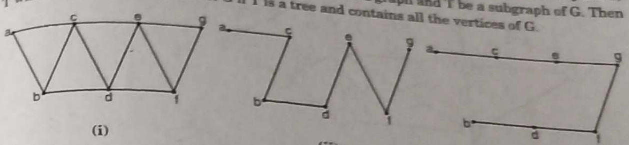
Converse : Let a simple graph G has a spanning tree.

To Prove : G is connected.

Let S be the spanning tree of G ∴ by definition, ∃ a path between any two vertices of G. Thus G is connected.

Q 41. What is minimum spanning tree of a graph? Write down Prim's and Kruskal's algorithms and execute them by taking a suitable example. **(PTU, May 2008)**

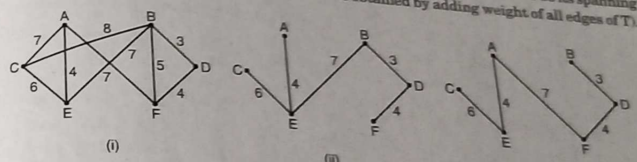
Solution. Spanning Tree : Let G be a connected graph and T be a subgraph of G. Then T will be called spanning tree of G if T is a tree and contains all the vertices of G.



(i) (ii) (iii)

Fig. (i) is a connected graph G and Fig. (ii) and (iii) are spanning trees of G.

Minimum Spanning Tree : Let G be a connected weighted graph and let T be its spanning tree with weight as small as possible. (weight of T is obtained by adding weight of all edges of T).



(i) (ii) (iii)

Fig. (i) is graph G and Fig. (ii) and (iii) are min. spanning trees with weight 24. Note that graph G has 6 vertices, therefore, min. spanning tree will have 5 edges.

Prim Algorithm : This algorithm is used for to find minimum spanning tree. Its algorithm is given as under :

I. Represent the distance network in matrix form.

II. Select the row 1 and delete the first column and mark the row 1 with*.

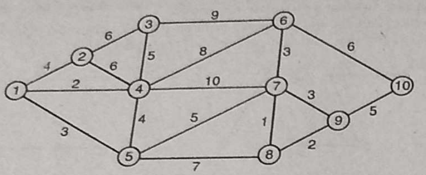
III. Select the minimum of the undeleted values among the rows and marking it by a square around it. Identify that column and mark also this row by * and delete that column.

IV. Check whether all columns are deleted if not then repeat the process otherwise show the arcs in the spanning tree corresponding to the cells of the matrix marked with squares by thick lines.

V. Find the total of all values (marked with square). This is the minimized total length of the edges to connect all the nodes of the network so for as minimum spanning tree concept.

This Algorithm can be easily understand by the following example.

e.g : Find the minimum spanning tree of the following graph as shown below by using prim algorithm.



Sol. Step I.

	1	2	3	4	5	6	7	8	9	10
1	–	4	∞	2	3	∞	∞	∞	∞	∞
2	4	–	6	6	∞	∞	∞	∞	∞	∞
3	∞	6	–	5	∞	9	∞	∞	∞	∞
4	2	6	5	–	4	8	10	∞	∞	∞
5	3	∞	∞	4	–	∞	5	7	∞	∞
6	∞	∞	9	8	∞	–	3	∞	∞	6
7	∞	∞	∞	10	5	3	–	1	3	∞
8	∞	∞	∞	∞	7	∞	1	–	2	∞
9	∞	∞	∞	∞	∞	∞	3	2	–	5
10	∞	∞	∞	∞	∞	∞	3	2	–	5
11	∞	∞	∞	∞	∞	6	∞	∞	5	–

Select first row and delete first column.

II

	2	3	4	5	6	7	8	9	10
*1	4	∞	[2]	3	∞	∞	∞	∞	∞
2	–	6	6	∞	∞	∞	∞	∞	∞
3	6	–	5	–	9	∞	∞	∞	∞
4	6	5	–	4	8	10	∞	∞	∞
5	∞	∞	4	–	∞	5	7	∞	∞
6	∞	9	8	∞	–	3	∞	∞	6
7	∞	∞	10	5	3	–	1	3	∞
8	∞	∞	∞	7	∞	1	–	2	∞
9	∞	∞	∞	∞	∞	3	2	–	5
10	∞	∞	∞	∞	6	∞	∞	5	–

Selecting the minimum of undeleted cell values marked with square and deleting that column and also marked row-4 with*.

III

	2	3	5	6	7	8	9	10
*1	4	∞	[3]	∞	∞	∞	∞	∞
2	–	6	∞	∞	∞	∞	∞	∞
3	6	–	∞	9	∞	∞	∞	∞
4	6	5	4	8	10	∞	∞	∞
5	∞	∞	–	∞	5	7	∞	∞
6	∞	9	∞	–	3	∞	∞	6
7	∞	∞	5	3	–	1	3	∞
8	∞	∞	7	∞	1	–	2	∞
9	∞	∞	∞	∞	3	2	–	5
10	∞	∞	∞	6	∞	∞	5	–

repeat the similar process as for previous table.

IV

	2	3	6	7	8	9	10
*1	[4]	∞	∞	∞	∞	∞	∞
2	–	6	∞	∞	∞	∞	∞
3	6	–	9	∞	∞	∞	∞
*4	6	5	8	10	∞	∞	∞
*5	∞	∞	∞	5	7	∞	∞
6	∞	9	–	3	∞	∞	6
7	∞	∞	3	–	1	3	∞
8	∞	∞	∞	1	–	2	∞
9	∞	∞	∞	3	2	–	5
10	∞	∞	6	∞	∞	5	–

V

	3	6	7	8	9	10
*1	∞	∞	∞	∞	∞	∞
*2	6	∞	∞	∞	∞	∞
3	–	9	∞	∞	∞	∞
*4	5	8	10	∞	∞	∞
*5	∞	∞	[5]	7	∞	∞
6	9	–	3	∞	∞	6
7	∞	3	–	1	3	∞
8	∞	∞	1	–	2	∞
9	∞	∞	3	2	–	5
10	∞	6	∞	∞	5	

VI

	3	6	8	9	10
*1	∞	∞	∞	∞	∞
*2	6	∞	∞	∞	∞
3	–	9	∞	∞	∞
*4	5	8	∞	∞	∞
*5	∞	∞	7	∞	∞
6	9	–	∞	∞	6
*7	∞	3	[1]	3	∞
8	∞	∞	–	2	∞
9	∞	∞	2	–	5
10	∞	6	∞	5	–

VII

	3	6	9	10
*1	∞	∞	∞	∞
*2	6	∞	∞	∞
3	–	9	∞	∞
*4	5	8	∞	∞
*5	∞	∞	∞	∞
6	9	–	∞	∞
*7	∞	3	3	∞
*8	∞	∞	[2]	∞
9	∞	∞	–	5
10	∞	6	5	–

VIII

	3	6	10
*1	∞	∞	∞
*2	6	∞	∞
3	–	9	∞
*4	5	8	∞
*5	∞	∞	∞
6	9	–	6
*7	∞	[3]	∞
*8	∞	∞	∞
*9	∞	∞	5
10	∞	6	–

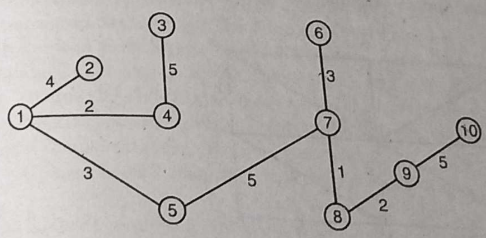
IX

	3	10
*1	∞	∞
*2	6	∞
3	–	∞
*4	5	∞
*5	∞	∞
*6	9	6
*7	∞	∞
*8	∞	∞
*9	∞	[5]
10	∞	–

X

	3
*1	∞
*2	6
3	–
*4	[5]
*5	∞
*6	9
*7	∞
*8	∞
*9	∞
*10	∞

So all the columns are deleted.
Hence minimum spanning tree is given as under :



and its length = 30.

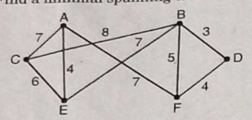
Kruskal Algorithm to find Minimum Spanning Tree : Input is a connected weighted graph G with n vertices.

Step 1. Arrange the edges of G in order of increasing weights.

Step 2. Starting only with the vertices of G and proceeding sequentially, add each edge which does not result in a cycle unti $n-1$ edges are added.

Step 3. Exit.

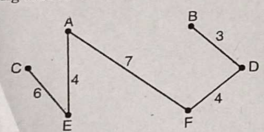
For example : Find a minimal spanning tree of a weighted graph given below.



Sol. First we order the edges by increasing weights and then we successively add edges without forming any cycles. Until five edges are included as folows :

Edges :	BD	AE	DF	BF	CE	AC	AF	BE	BC
Weight :	3	4	4	5	6	7	7	7	8
Add? :	Yes	Yes	Yes	No	Yes	No	Yes		

∴ Minimal spanning tree of G is obtained containing the edges BD, AE, DF, CE, AF.

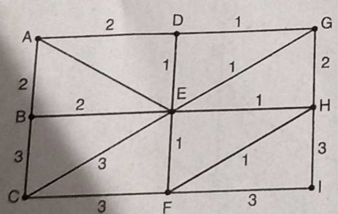


Kruskal algorithm : Let G be a connected weighted graph with n-vertices.

I. Arrange the edges of G in order of decreasing weights.

II. Remove the edges sequentially one by one so that the grap is not disconnected with $n-1$ edges remain.

For example : Find the minimal spanning tree of a weighted graph given below :

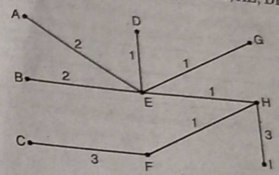


Sol. First of all we arrange the edges in order of decreasing weights. Then we remove the

edges successively one by one so that the graph is not disconnected untill we have $9-1=8$ edges remains.

Edges :	CE	BC	CF	FI	HI	AB	BE	AE	AD	GH	DG	DE	EG	EH	EF	FH
Weight :	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1
Delete :	Yes	Yes	No	Yes	No	Yes	No	No	Yes	Yes	Yes	No	No	No	Yes	No

So minimal spanning tree containing edges. CF, HI, BE, AE, DE, EG, EH, FH



Q 42. If a graph G has more than two vertices of odd degree, then prove that there can be no Euler path in G.

(PTU, May 2008)

Solution. Since G has more than two vertices of odd degree and let v_1, v_2, v_3 be such vertices in G. If there is a Euler path in G then it must have (arrive) each of the vertices v_1, v_2, v_3 with no may to return (or leave) one of the vertices say v_1 may be beginning and another vertex say v_2 at the end of an Euler path in G but this leaves the vertex v_3 at one end of an untravelled edge. Thus there can be no Euler path in G.

Q 43. Let G = (V, E) be an undirected graph with k-components and $|V| = n$, $|E| = m$. Prove that $m \geq n - k$.

(PTU, Dec. 2009)

Solution. We will use the second principle of induction (strong induction) for m.

Induction Basis : $m = 0$. The components are trivial and $n = k$.

Induction Hypothesis : The theorem is true for $m < p$. $(p \geq 1)$

Induction Statement : The theorem is true for $m = p$.

Induction Statement Proof : We choose a component G_1 of G which has at least one edge. We label that edge e and the end vertices u and v. We also label G_2 as the subgraph of G and G_1, obtained by removing the edge e from G_1 (but not the vertices u and v). We label G' as the graph obtained by removing the edge e from G (but not the vertices u and v) and let k' be the number of components of G'. We have two cases :

1. G_2 is connected. Then, $k' = k$. We use the Induction Hypothesis on G' :

$$n - k = n - k' \leq m - 1 < m.$$

2. G_2 is not connected. Then there is only one path between u and v :

u, e, v

and no other path. Thus, there are two components in G_2 and $k' = k + 1$. We use the Induction Hypothesis on G'.

$$n - k' = n - k - 1 \leq m - 1.$$

Hence $n - k \leq m$.

Q 44. Explain any two applications of Coloring of a Graph. **(PTU, Dec. 2009)**

Solution. Graph Coloring : Applications

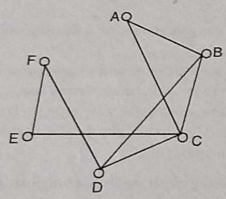
Let's see how this information about graphs and coloring can be used to solve real life problems :

A tropical fish hobbist had six different types of fish : **Alphas, Betas, Certas, Deltas, Epsalas and Fetas,** which shall hence forth be designated by **A, B, C, D, E, and F,** respectively. Because of predator-prey relationships, water conditions, and size, some fish can be kept in the same tank. The following table shows which fish cannot be together :

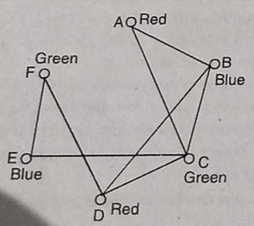
Type	A	B	C	D	E	F
Cannot be with	**B, C**	**A, C, E**	**A, B, D, E**	**C, F**	**B, C, F**	**D, E**

What is the smallest number of tanks needed to keep all the fish?

We will use a graph to help us answer this question. Below you will see an uncolored graph that describes this situation.



Here is the graph again -- now with color!!



If each vertex represents one of the types of fish and each edge connects vertices that are not compatible.

Make a list of the number of tanks that are needed and the kinds of fish that will be kept in each one.

Several different combinations of fish are possible depending on how the graph is colored. Below is the solution for how our graph was colored. Note that fish with vertices of the same color go into the same tank.

Tank 1	Tank 2	Tank 3
Alphas and Deltas	**Fetas and Certas**	**Betas and Epsalas**

Thus, he fewest number of tanks the tropical fish owner will need is three.

2. Scheduling : Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in conflict in the sense that they may not be assigned to the same time slot, for example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum makespan, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircrafts to flights, the resulting confict graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

3. Register allocation : A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the compiled program are kept in the fast processor registers. Ideally, values are assigned to registers so that they can all reside in the registers when they are used.

The textbook approach to this problem is to model it as a graph coloring problem. The compiler constructs an interference graph, where vertices are symbolic registers and an edge connects two nodes if they are needed at the same time. If the graph can be colored with k colors then the variables can be stored in k registers.

Q 45. Consider the following relation on the set A = {1, 2, 3, 4} : R = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)}. Draw its diagraph. Is R (i) reflexive (ii) antisymmetric and (iii) transitive? **(PTU, Dec. 2010)**

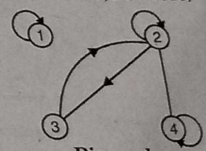
Solution.

Given $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$

(i) as $(3, 3) \notin R$ $\therefore$ R is not reflexive.

(ii) As $(2, 3), (3, 2) \in R$ but $2 \neq 3$ $\therefore$ R is not antisymmetric.

(iii) As $(4, 2), (2, 3) \in R$ but $(4, 3) \notin R$ $\therefore$ R is not transitive.



Diagraph

Q 46. Show that the sum of degree of all the vertices in a graph is even. **(PTU, Dec. 2010)**

Ans. Degree of vertex : Degree of a vertex v in a graph G is the number of edges incident on v, written as deg (v).

Since, while counting the degree of vertices of G, each edge is counted twice.
∴ The sum of degree of all the vertices in a graph is even.
Hence the result.

Q 47. Show that the chromatic number of a graph C_n, where C_n is the cyclic with n vertices is either 2 or 3. **(PTU, Dec. 2010)**

Solution. Case I. If n is odd and $n > 1$

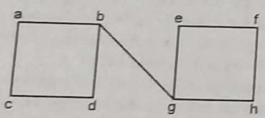
If in C_n, n is odd then we will take up an initial vertex first we will use only two colours and alternate colours as graph is traversed in clockwise direction. However the nth vertex reached is adjacent to two vertices of different colours, the first and $(n-1)$th. Hence a third colour is needed.

Case II. If n is even,

If in C_n, n is even. Then we assign a colour p_1 to initial vertex. We will traverse the graph in clockwise direction colouring the second vertex with p_2 colour then third with p_1 and so on. The nth vertex is coloured with p_2 color since both vertex adjacent to it are of p_1 colour since n is even.

Therefore we conclude from the both cases that, the chromatic number of graph C_n where C_n is a cycle with n vertices is either 2 or 3.

Q 48.



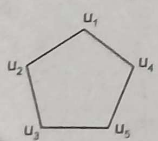
How many different paths are there between vertices a and h in the above graph? How many of these paths have length 5? **(PTU, Dec. 2011)**

Solution. (i) abgh (ii) abgefh (iii) acdbgh (iv) acdbgefh

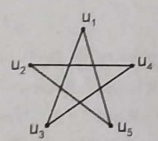
The path (ii) and (iii) is of length 5.

Q 49. Define the term, 'Complement of a graph' and give an example. **(PTU, Dec. 2011)**

Solution. The term compliment of graph can be explaned with the help of an example. e.g. In graph G, u_1 is joined to u_3 and u_5. So in complement of G, u_1 is joined to u_2 and u_4. In G_1 we have u_2 is joined to u_4 and u_5 and in complement of G u_2 is joined to u_1 and u_3 similarly for other vertices.



Graph 'G'



Compliment of 'G'

Q 50. Show that the sum of the degrees of the vertices of a non directed graph is [twice the] number of edges in the graph. **(PTU, May 2012)**

Solution. Let G be a non-directed graph with n-vertices $\{v_1, ... v_n\}$ and every edge contribute two to the sum i.e. one from L.H.S. and one from R.H.S. ∴ The sum of the degree of

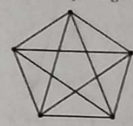
all vertices in G is twice the number of edges of G i.e. $\sum_{i=1}^{n} \deg_G(v_i) = 2|E|$. Let G be the directed graph and say (v_1, v_2) be the edge, this edge contributes one to the outdegree of v_1 and one to the indegree v_2 and this is true for all edges.

∴ Sum of outdegrees of G = Sum of all indegree of G = $|E|$.

Thus, the sum of the degrees of the vertices of a non-directed graph is twice the number of edges in the graph.

Q 51. Give an example of a graph which is non-planar. **(PTU, Dec. 2011)**

Solution. The graph for K_5 i.e. complete graph with 5-vertices is given as under :



$|V| = 5$, $|E|$ = no. of edges = 10

If G is planer then $|E| \leq 3|V| - 6$

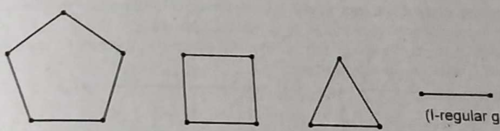
i.e. $10 \leq 3.5 - 6 \Rightarrow 10 \leq 9$, a contradiction

∴ K_5 is non-planar.

Q 52. Define the terms (i) Regular graph (ii) Complete graph. **(PTU, May 2012, 2010)**

Solution. (i) Regular Graph : A graph is said to be regular if every vertex of it has same degree. If each vertex has degree k, then k-regular.

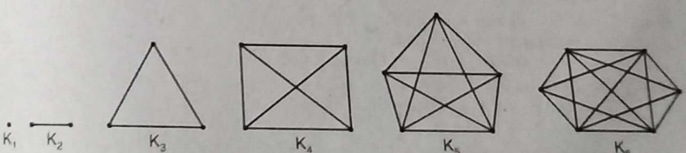
For example :



2-regular graphs

(ii) Complete Graph : If every vertex of a graph is connected with every other vertex of that graph, then graph is complete graph. A complete graph with n vertices is denoted by K_n.

For example :



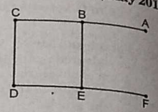
Q 53. Give an example of a graph that has neither an Euler circuit nor a Hamiltonian circuit. **(PTU, May 2012)**

Solution. Here every vertex is not of even degree as deg (B) = deg (E) = 3

$\therefore$ The given graph is not Euler circuit.

Also there is no edge between A and F

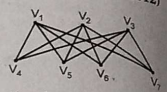
$\therefore$ It is not a Hamiltonian circuit.



Q 54. Find the chromatic number of the complete bipartite graph $K_{3,4}$. **(PTU, Dec. 2012)**

Solution.

Given vertices	V_1	V_2	V_3	V_4	V_5	V_6	V_7
Decreasing order of degree	4	4	4	3	3	3	3
Colors	C_1	C_1	C_1	C_2	C_2	C_2	C_2



We color the vertices in such a manner that, no two adjacent vertices having same color.

$\therefore$ all vertices are colored and 2 colors are used $\therefore X(G) \leq 2$.

Further $V_1 V_4$ are adjacent $\therefore$ atleast 2 colors are required.

Thus chromatic number of $K_{3,4}$ is 2.

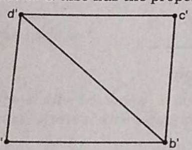
Q 55. List any five properties of a graph which are invariant under graph isomorphism. **(PTU, Dec. 2012)**

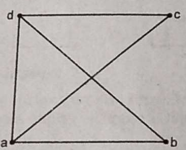
Solution. Some graphs have some structure i.e. same vertices and edges but differ only in way of representation. This property of commonness is called isomorphism.

A graph $G_1 (V_1, E_1)$ is isomorphic to $G_2 (V_2, E_2)$ if there is $1-1$ correspondence between the edges E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 with end vertices u_2 and v_2 in G_2 which corresponds to u_1 and v_1 respectively. Such a pair of correspondenceness is called graph isomorphism.

We say a properly of graphs is a graph invariant if whenever a graph G has a property, any graph isomorphic to G also has the property.

e.g. :





Here $V_1 = \{a, b, c, d\}$ and $V_2 = \{a', b', c', d'\}$

we define a mapping $f : V_1 \to V_2$ s.t.

$f(a) = b'$; $f(b) = c'$; $f(c) = a'$; $f(d) = d'$

Here the mapping is $1-1$ onto

$(a, b) \in E_1 \Rightarrow \{f(a), f(b)\} = (b', c') \in E_2$

$(a, c) \in E_1$ $(b', a') \in E_2$

$(a, d) \in E_1$ $(b', d') \in E_2$

$(b, d) \in E_1$ $(c', d') \in E_2$

$(c, d) \in E_1$ $(a', d') \in E_2$

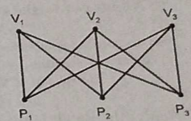
$\therefore G_1 (V_1, E_1)$ is isomorphic to $G_2 (V_2, E_2)$.

Thus the properties of graph which are invariant under graph isomorphism

(i) number of vertices
(ii) number of edges
(iii) an equal number of vertices with a given degree (degree sequence)
(iv) no. of connected components
(v) has a circuit of equal length
(vi) number of loops
(vii) no. of set of parallel edges
(viii) has a Hamiltonian circuit
(ix) G contains r complete graph K_n
(x) G contains r complete bipartite graphs $K_{m,n}$.

Q 56. Prove that it is not possible be supply three utilities to three places by conduits without crossing over. **(PTU, Dec. 2012)**

Solution. The graph $K_{3,3}$ given as under :



where V_1, V_2, V_3 are three utilises and P_1, P_2 and P_3 are three places.

$$|V| = 6, |E| = 9$$

Now $K_{3,3}$ is planer if $|E| \leq 3|V| - 6$

$\Rightarrow 9 \leq 3.6 - 6$ *i.e.* $9 \leq 12$ which is true.

but it is clear from the graph that each region is bounded by atleast four edges.

$2|E| \geq 4|R|$

i.e. $2|E| \geq 4[|E| + 2 - |V|] \Rightarrow 2|E| \geq 4[9 + 2 - 6]$

$\Rightarrow 2.9 \geq 4.5 \Rightarrow 18 \geq 20$, a contradiction.

$\therefore K_{3,3}$ is non-planer.

So it is not possible be supply three utilities to three places by conduits without crossing over.

Q 57. Prove that an undirected graph has an even number of vertices with odd degree. **(PTU, May 2013)**

Solution. Let G (V, E) be the undirected graph. Since every edge is incident with exactly 2 vertices $\therefore$ while counting the degree of vertices of G, each edge is counted twice. $\therefore$ The sum of degree of all vertices in a graph is even

i.e. $$\sum_i \deg(v_i) = \text{even} \quad ....(1)$$

Let V_1 and V_2 be the set of vertices of G of even and odd degrees respectively

$$\sum_{v_i \in V_1} \deg(v_i) + \sum_{v_i \in V_2} \deg(v_i) = \text{even}$$

$$\therefore \quad \sum_{v_i \in V_2} \deg(v_i) = \text{even} - \sum_{v_i \in V_1} \deg(v_i) \quad ...(2)$$

Since each $\deg(v_i)$ is even $\therefore$ from (2) ; we have

$$\sum_{v_i \in V_2} \deg(v_i) = \text{even} - \text{even} = \text{even}$$

Thus, an undirected graph has an even number of vertices with odd degree.

□□□

Q 58. Let G be a connected planar graph with p vertices and q edges, where $p \geq 3$. Then prove that $q \leq 3p - 6$. **(PTU, Dec. 2013)**

Solution. For proving this result, first of all we prove Euler's formula

Euler's Formula : Let G (V, E, R) be a planar graph with V, E and R as number of vertices, edges and regions respectively then $V - E + R = 2$.

Proof : We shall prove the result by induction on R.

$$\text{Let } R = 1 \Rightarrow E = 0 \text{ and } V = 1 \therefore V - E + R = 1 - 0 + 1 = 2$$

$\therefore$ The result is true for $R = 1$.

Let us assume the result for $R = m$, $m \in N$

$\therefore$ We want to prove the result for $R = m + 1$.

On removing an edge now which will be common to the boundary of two regions, we obtain a new graph G' (V', E', R') which has m regions now.

For m regions, the result is true. $\therefore V' - E' + R' = 2$ ...... (1)

Since, we have obtained G' by removing an edge.

$\therefore$ Number of vertices remain same.

$\therefore$ $V = V'$ and $E' = E - 1$ and $R' = R - 1$ ...... (2)

(as on removing common edge between two regions, we get one region only)

$\therefore$ for m + 1 regions,

$$V - E + R = V' - (E' + 1) + (R' + 1) \quad \text{[using (2)]}$$

$$= (V' - E' + R') - 1 + 1 = 2 \quad \text{[using (1)]}$$

Hence, the result is true for $R = m + 1$ therefore by induction $V - E + R = 2$.

Here G be a connected planner graph with p vertices and q edges

í Euler formula becomes, $p - q + R = 2$ .....(3)

Next we want to prove that $2\,|q| \leq 3\,|R|$

Since $|q| > 1$, and if G has unbdd region then $|R| = 1$ and $|q| \geq 2$. Then given result holds obviously.

If $R > 1$, each region R is bdd by atleast three edges and in planar connected graph each edge touches atmost two region, $2\,|q| \geq 3\,|R|$ holds good.

$$\Rightarrow |R| \leq \frac{2}{3}\,|q| \Rightarrow |p| + |R| \leq \frac{2}{3}\,|q| + |p|$$

Also by Euler's formula we have $|p| - |q| + |R| = 2$

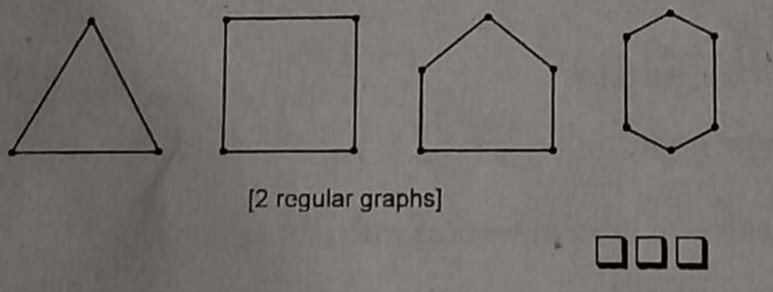
From (3) & (4), we have

$$2 + |q| \leq \frac{2}{3}\,|q| + |p| \Rightarrow \frac{1}{3}\,|q| \leq |p| - 2$$

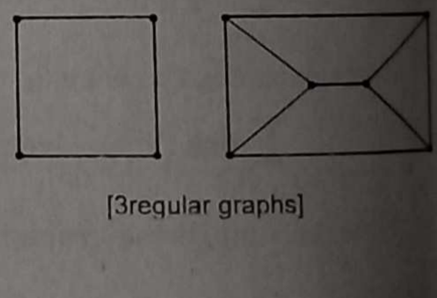
$$\Rightarrow |q| \leq 3\,|p| - 6$$

Q 59. Draw regular graphs of degree 2 and 3. **(PTU, Dec. 2014)**

Solution. Regular graph : A graph (simple) is said to be regular if every vertex of it have same degree. If the degree of each vertex is n then the graph is called n-regular graph.



[2 regular graphs]



[3regular graphs]

□□□